Theoretical physics V
Sheet 13

SoSe 2025 **2.07.2025**

Exercise 1 *Second quantization and the Schrödinger equation*

Let us consider the N-body wave function defined as

$$\Psi_E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}} \langle 0|\hat{\psi}(\boldsymbol{r}_1) \ldots \hat{\psi}(\boldsymbol{r}_N)|E, N\rangle \,, \tag{1}$$

where $|E, N\rangle$ is an N-particles energy eigenstate with eigenvalue E of the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \int \mathrm{d}^3\boldsymbol{r} \hat{\psi}^\dagger(\boldsymbol{r}) \nabla^2 \hat{\psi}(\boldsymbol{r}) + \frac{1}{2} \int \mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{r}_2 \hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}^\dagger(\boldsymbol{r}_2) v(\boldsymbol{r}_1, \boldsymbol{r}_2)\hat{\psi}(\boldsymbol{r}_2)\hat{\psi}(\boldsymbol{r}_1). \tag{2}$$

a) Show that

$$E\Psi_E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}} \langle 0|\hat{\psi}(\boldsymbol{r}_1) \ldots \hat{\psi}(\boldsymbol{r}_N)\hat{H}|E, N\rangle. \tag{3}$$

(1 point)

b) Show that the wave function satisfies the N-particles Schrödinger equation

$$\left[-\sum_{i=1}^{N} \frac{\hbar^2}{2m}\nabla_i^2 + \sum_{i<j} v(\boldsymbol{r}_i, \boldsymbol{r}_j)\right] \Psi_E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = E\Psi_E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N). \tag{4}$$

(2 points)

Exercise 2 *Real Scalar Field*

Consider the Klein-Gordon equation for the scalar field $\phi(\vec{r}, t)$:

$$\partial_\mu \partial^\mu \phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \,, \tag{5}$$

with m the rest mass.

a) Show that Eq. (5) is the Euler-Lagrange equation of the Lagrange density

$$\mathcal{L}(x) = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - \frac{m^2c^2}{\hbar^2}\phi^2\right) = \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial}{\partial t}\phi\right)^2 - |\vec{\nabla}\phi|^2 - \frac{m^2c^2}{\hbar^2}\phi^2\right] \tag{6}$$

(1 point)

b) Use the canonically conjugated momentum field $\Pi = (1/c)\partial\phi/\partial t$ and show that the Hamiltonian density takes the form

$$\mathcal{H}(x) = \Pi \frac{1}{c}\frac{\partial \phi}{\partial t} - \mathcal{L}(x) = \frac{\Pi^2}{2} + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2 \,. \tag{7}$$

(1 point)

c) Assume a box of volume Ω with periodic boundary conditions. Expand the field ϕ in a Fourier series

$$\phi(\vec{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} q_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}} \tag{8}$$

and show that the Hamiltonian $H = \int \mathrm{d}\vec{r} \mathcal{H}(\vec{r})$ takes the form

$$H = \sum_{\vec{k}} \frac{\Pi_{\vec{k}}^2}{2} + \frac{\omega_{\vec{k}}^2 |q_{\vec{k}}|^2}{2} \,, \tag{9}$$

with $\Pi_{\vec{k}} = (1/c)\partial q_{\vec{k}}/\partial t$. *(2 points)*

d) We now define

$$q_{\vec{k}}(t) = \sqrt{\frac{\hbar}{2\omega_{\vec{k}}}} \left[\hat{a}_{\vec{k}}\, e^{-i\omega_{\vec{k}}t} + \hat{a}^\dagger_{-\vec{k}}\, e^{i\omega_{\vec{k}}t} \right] \,, \tag{10}$$

with the operators $\hat{a}_{\vec{k}}$ obeying the commutation relations $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'}$ and $[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0$. Using this definition, bring the Hamiltonian into the form

$$\hat{H} = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \frac{1}{2} \right) \,, \tag{11}$$

and show that

$$\omega_{\vec{k}} = \sqrt{c^2|\vec{k}|^2 + m^2c^2} > 0 \,. \tag{12}$$

(2 points)