Theoretical physics V
Sheet 14

SoSe 2025 **9.07.2025**

Exercise 1 *Fermions on a lattice*

Let us consider a one-dimensional lattice made of N sites. A simple model for a metal consists in allowing non-interacting electrons to jump from one site of the lattice to a neighbouring one. The Hamiltonian for such a model reads

$$\hat{H} = -t \sum_{j=1}^{N} (\hat{c}_{j+1}^{\dagger} \hat{c}_j + \hat{c}_j^{\dagger} \hat{c}_{j+1}), \tag{1}$$

where operator $\hat{c}_j$ destroys an electron on site j, and follows anticommutation relations: $\{\hat{c}_j, \hat{c}_{j'}^{\dagger}\} = \delta_{jj'}$ and $\{\hat{c}_j, \hat{c}_{j'}\} = 0$. We assume that the crystal has periodic boundaries, such that $\hat{c}_{N+1} = \hat{c}_1$.

a) Defining the Fourier transform of $\hat{c}_j$ as $\hat{c}_k = (1/\sqrt{N}) \sum_{j=1}^{N} \hat{c}_j \exp(ikj)$, with waves vectors $k = 0, 2\pi/N, \ldots, 2\pi n/N, \ldots, 2\pi(N-1)/N$, show that the Hamiltonian in Fourier space reads

$$\hat{H} = \sum_{k} \epsilon(k) \hat{c}_k^{\dagger} \hat{c}_k, \tag{2}$$

where $\epsilon(k) = -2t \cos(k)$.

Hint: We give the identity:

$$\sum_{j=1}^{N} e^{i(k-k')j} = N \delta_{k,k'}$$

(2 points)

b) Let us assume that the lattice is at half-filling, namely for a chain of N sites, there are $N/2$ electrons. Determine the ground state energy and the ground state.

(1 point)

c) Given the initial state $|\psi_0\rangle = \hat{c}_{j_0}^{\dagger}|0\rangle$, determine the evolution of the local number of electrons $n_j(t)$ such that

$$n_j(t) = \langle \psi_0 | \hat{c}_j^{\dagger}(t) \hat{c}_j(t) | \psi_0 \rangle. \tag{3}$$

Hint: You shall treat this problem in Heisenberg picture.

(2 points)

Exercise 2 *Microcausality*

In the following, we will see how the concept of causality naturally emerges from the Klein-Gordon equation and can be extracted from the commutation relation of the quantum field.

a) Show that the group velocity $v_g = \frac{\partial \omega_k}{\partial |k|}$ of a wavepacket described by a Klein-Gordon equation cannot exceed the speed of light.

(1 point)

b) Let us consider the quantum field $\phi(x) = \phi(\boldsymbol{r}, t)$, which is a solution of the Klein-Gordon equation. Show that for a fixed point of space-time y the commutator $[\phi(x), \phi(y)]$ is also a solution of the Klein-Gordon equation. Justify then that

$$[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle \equiv i\Delta(x - y)\,. \tag{4}$$

Hint: You may use that $[\phi(\boldsymbol{r}, t), \phi(\boldsymbol{r'}, t)] = 0$ is a c-number, namely a multiple of the identity operator.

(2 points)

c) By the means of the Fourier decomposition of the field,

$$\phi(\boldsymbol{r}, t) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} + \hat{a}^\dagger_{\boldsymbol{k}} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)}\right]\,, \tag{5}$$

determine the form of $\Delta(x)$ and discuss the case when $x^2 = c^2t^2 - |\boldsymbol{r}|^2 < 0$.

(3 points)