

Exercises for Theoretical Physics V

SoSe 2026

Sheet 3

22.04.2026

Exercise 4 *Properties of the Dirac matrices*

Given the Dirac equation

$$\frac{1}{c} \frac{\partial}{\partial t} \psi(\mathbf{r}, t) + \sum_{j=1}^3 \alpha^j \frac{\partial}{\partial x^j} \psi(\mathbf{r}, t) + i\beta \frac{mc}{\hbar} \psi(\mathbf{r}, t) = 0,$$

show that the choice

$$\alpha^j = \begin{pmatrix} \mathbf{0}_2 & \sigma^j \\ \sigma^j & \mathbf{0}_2 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad (1)$$

where σ^j are the Pauli matrices while $\mathbf{1}_n$ and $\mathbf{0}_n$ are respectively the $n \times n$ identity and zero matrices, satisfy the following requirements:

a) $(\alpha^j)^2 = \mathbf{1}_4$ and $\beta^2 = \mathbf{1}_4$

(2 points)

b) $\{\alpha^j, \beta\} = 0$ and $\{\alpha^j, \alpha^k\} = 0$ for $j \neq k$ where $\{A, B\} = AB + BA$.

(2 points)

Exercise 5 *Transformation between inertial reference frames*

As seen in the lectures, a change of reference frame leads to a transformation of the Dirac spinor $\psi' = S\psi$. For the transformation S to leave the Dirac equation invariant, it needs to fulfill the following conditions

$$S^\dagger \alpha^\mu S = \Lambda^\mu{}_\nu \alpha^\nu \quad \text{and} \quad S^\dagger \beta S = \beta, \quad (2)$$

where Λ is a transformation of the Lorentz group. The α and β matrices are defined as $\alpha^0 = \mathbf{1}_4$ and in Eq. (1).

Show that the operators S given in the following fulfill the conditions stated in Eq. (2) for the corresponding reference frame transformations.

a) Consider a rotation around the z axis by an angle θ , the coordinates transform as:

$$\begin{aligned} ct' &= ct; \quad z' = z; \\ x' &= \cos \theta x + \sin \theta y; \quad y' = \cos \theta y - \sin \theta x. \end{aligned}$$

Identify the matrix elements of Λ and show that the transformation

$$S = \cos \frac{\theta}{2} \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} + i \sin \frac{\theta}{2} \begin{pmatrix} \sigma^z & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma^z \end{pmatrix}$$

fulfills the conditions stated in Eq. (2).

(2 points)

- b) We consider now a Lorentz boost along the x axis by a rapidity χ that transforms the coordinates as:

$$ct' = \cosh \chi ct - \sinh \chi x ; x' = \cosh \chi x - \sinh \chi ct ; \\ y' = y ; z' = z .$$

Determine the elements of the matrix Λ and show that the transformation

$$S = \cosh \frac{\chi}{2} \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} - \sinh \frac{\chi}{2} \begin{pmatrix} \mathbf{0}_2 & \sigma^x \\ \sigma^x & \mathbf{0}_2 \end{pmatrix} ,$$

fulfills the conditions stated in Eq. (2).

(2 points)

- c) Let us finally consider a spatial reflection given by the transformation

$$ct' = ct ; x'_i = -x_i .$$

Identify the form taken by the matrix Λ and show that the transformation

$$S = \beta$$

fulfills the conditions stated in Eq. (2).

(1 point)

Exercise 6 *Non-relativistic limit of the Klein-Gordon equation*

Let us consider a scalar boson described by the wave function $\phi(\mathbf{r}, t)$. The behaviour of the wave function is governed by the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \phi(\mathbf{r}, t) = 0 ,$$

where $\partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ stands for the d'Alembert operator and $\eta^{\mu\nu}$ is the flat metric tensor of special relativity, a diagonal matrix with coefficients $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

- a) Assuming that the wave function has the form $\phi(\mathbf{r}, t) = e^{-iEt/\hbar} \phi_0(\mathbf{r})$, show that the behaviour of the wave function is described by the equation

$$E^2 \phi(\mathbf{r}, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \phi(\mathbf{r}, t) . \quad (3)$$

Show explicitly that Eq. (3) is fulfilled for $\phi_0(\mathbf{r}) = \exp(i\mathbf{p} \cdot \mathbf{r}/\hbar)/N$, where $\mathbf{p} \in \mathbb{R}^3$ and N is a normalization constant. Subsequently, verify that \mathbf{p} and E/c , as given by Eq. (3), are valid components of the energy-momentum four vector.

(2 points)

- b) Show that in the non-relativistic case, namely in the limit where $E = E_{NR} + mc^2$ with $mc^2 \gg E_{NR}$, the Klein-Gordon equation is approximated by the Schrödinger equation of a free particle:

$$E_{NR} \phi(\mathbf{r}, t) \simeq -\frac{\hbar^2}{2m} \nabla^2 \phi(\mathbf{r}, t) .$$

(1 point)