Exercises for
Theoretical Physics V

SoSe 2026 **Sheet 4** **29.04.2026**

Exercise 7 *Properties of the gamma matrices*

The Dirac Equation in the van-der-Waerden form is given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\psi(\boldsymbol{r},t) = 0, \tag{1}$$

where the γ-matrices are defined by the general expression $\gamma^\mu = \beta\alpha^\mu$, namely $\gamma^0 = \beta$, while $\gamma^j = \beta\alpha^j$ for $j = 1, 2, 3$. Show the following properties, without explicitly writing the γ-matrices:

a) Anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, with the flat space-time metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1 point)

b) The matrices γ^j are anti-hermitian.

(1 point)

c) The trace of the matrices γ^μ vanishes.

(1 point)

Exercise 8 *Dirac spinor and Lorentz boost*

The solutions for the Dirac equation for a free particle of mass m moving with momentum $\boldsymbol{p}$ belong to a Hilbert space spanned by the basis $\psi^{(j)}$ whose elements are defined as:

$$\psi^{(j)}(\boldsymbol{x},t) = \mathcal{N}u^{(j)}(\boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{x}\ -Et)/\hbar} \text{ with } \mathcal{N} = \sqrt{(|E| + mc^2)/(2|E|V)}, \tag{2}$$

where the spinors $u^{(j)}$ take the form

$$u^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ -\dfrac{cp_z}{E+mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{cp_z}{|E|+mc^2} \\ -\dfrac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{|E|+mc^2} \\ \dfrac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with p_x, p_y, p_z being the components of the momentum $\boldsymbol{p}$, while E stands for the energy of the particle and V for the volume available to the particle.

a) We have previously seen that Dirac spinors transform as $\psi' = S\psi$ for a change of reference frame. Transform the solutions for the free electron Dirac equation in the case of $\boldsymbol{p} = p\boldsymbol{e}_x$ to a moving reference frame with $v = \beta c$, where $\beta = cp/E$. The Lorentz transformation for a boost in the x-direction is given by

$$S_L = \cosh\frac{\chi}{2}\mathbf{1}_4 - \alpha_x \sinh\frac{\chi}{2}, \tag{3}$$

for $\tanh\chi = \beta$. Compare these results with the solutions of the electron at rest. Check whether the states are normalized.

Hint: Impose $\cosh(\chi) = E_{\boldsymbol{p}}/(mc^2)$ and $\sinh(\chi) = pc/(mc^2)$. Don't forget to apply the Lorentz transformation to the spacetime coordinates $(\boldsymbol{x}, t) \to (\boldsymbol{x}', t')$ as well!

(2 points)

b) Show that positive- (negative-) energy solutions of the Dirac equation for $\boldsymbol{p} = 0$ transform via a Lorentz boost into another positive- (negative-) energy solution of the Dirac equation.

Hint: Use the spinor transformation to a reference frame of rapidity χ along the direction $-\boldsymbol{n}$:

$$S_L = \cosh\frac{\chi}{2}\mathbf{1}_4 + \boldsymbol{n}\cdot\boldsymbol{\alpha}\sinh\frac{\chi}{2}.$$

(2 points)