Exercises for
Theoretical Physics V

SoSe 2026 **Sheet 5** **06.05.2026**

Exercise 9 *Charge conjugation and antiparticles*

We remind that the solutions for the Dirac equation for a free particle of mass m moving with momentum $\boldsymbol{p}$ take the form $\psi^{(j)}$ such that:

$$\psi^{(j)}(\boldsymbol{x},t) = \mathcal{N} u^{(j)}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x}\ -Et)/\hbar} \text{ with } \mathcal{N} = \sqrt{(|E|+mc^2)/(2|E|V)}, \tag{1}$$

where the spinors $u^{(j)}$ take the form

$$u^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ -\dfrac{cp_z}{E+mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{cp_z}{|E|+mc^2} \\ -\dfrac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{|E|+mc^2} \\ \dfrac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with E standing for the energy of the particle and V for the volume available to the particle. Let us consider the charge conjugate of the negative-energy solutions of the Dirac equations, namely

$$\phi(\boldsymbol{x},t) = \mathcal{C}(\psi^{(3,4)}(\boldsymbol{x},t))^*$$

where $\mathcal{C} = -i\beta\alpha^2$ with

$$\alpha^i = \begin{pmatrix} \mathbf{0}_2 & \sigma^i \\ \sigma^i & \mathbf{0}_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

a) Explicitly compute the form of ϕ for both cases $j = 3, 4$ and compare them to the positive-energy solutions $\psi^{(1,2)}(\boldsymbol{x},t)$ of the Dirac equation. *(2 points)*

b) Show that $[\mathcal{C},\alpha^1] = [\mathcal{C},\alpha^3] = 0$ (commutation). *(1 point)*

c) Show that $\{\mathcal{C},\beta\} = \{\mathcal{C},\alpha^2\} = 0$ (anticommutation). *(1 point)*

d) Show that $\mathcal{C}^* = \mathcal{C}$. *(1 point)*

e) Show that $\mathcal{C}^\dagger = \mathcal{C}$. *(1 point)*

f) Show that $\mathcal{C}^2 = \mathbf{1}_4 = \mathcal{C}^\dagger\mathcal{C}$. *(1 point)*

Exercise 10 *Helicity conservation*

The helicity operator is defined as $h = \boldsymbol{\Sigma} \cdot \boldsymbol{p}$, with the momentum operator $\boldsymbol{p} = (p_1, p_2, p_3)$ and the spin operator $\boldsymbol{\Sigma} = (\Sigma^1, \Sigma^2, \Sigma^3)$, with components

$$\Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}.$$

Here, σ^j, with $j = 1, 2, 3$, denote the Pauli matrices. The Hamiltonian is

$$H = c\,\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2 \tag{3}$$

and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$. The matrices α^j and β are explicitly given in Eq. (2).

a) Show that the helicity is a conserved quantity of motion, namely that

$$[H, h] = 0.$$

(2 points)

b) Determine the eigenstates $\tilde{\psi}^{(j)}(\boldsymbol{x}, t)$ of the helicity operator h and compute their corresponding eigenvalues $h^{(j)}$ such that

$$h\tilde{\psi}^{(j)}(\boldsymbol{x}, t) = h^{(j)}\tilde{\psi}^{(j)}(\boldsymbol{x}, t).$$

Hint: since the helicity h commutes with the Dirac Hamiltonian H, the eigenstates of h are superpositions of the degenerate pairs of the eigenstates given in Eq. (1). *(2 points)*