Exercises for
Theoretical Physics V

SoSe 2026 **Sheet 6** **13.05.2026**

Exercise 11 *Hydrogen atom in Dirac theory*

Consider the eigenstate problem of the Dirac equation for the hydrogen atom:

$$E\psi = H\psi,$$

where the Hamiltonian H is given by

$$H = -i\hbar c\,\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + mc^2\beta - \frac{e^2}{r}\mathbf{1}_4 \ .$$

with

$$\alpha^i = \begin{pmatrix} \mathbf{0}_2 & \sigma^i \\ \sigma^i & \mathbf{0}_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The total angular momentum is

$$\boldsymbol{J} = \frac{\boldsymbol{L}}{\hbar}\mathbf{1}_4 + \frac{\boldsymbol{\Sigma}}{2}$$

where $\boldsymbol{\Sigma} = (\Sigma^1, \Sigma^2, \Sigma^3)$, with $\Sigma^i = \begin{pmatrix} \sigma^i & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma^i \end{pmatrix}$, and the orbital angular momentum is

$$\boldsymbol{L} = -i\hbar \boldsymbol{r} \times \boldsymbol{\nabla} \ .$$

a) Show that $[H, \beta] = -2i\hbar c\,(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla})\beta$ and hence that

$$-\frac{1}{2}[H, \beta] = i\hbar c\,(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla})\beta \ .$$

Hint: Use $\{\alpha^i, \beta\} = \mathbf{0}_4$. *(1 point)*

b) Show that $[\beta, \Sigma^i] = \mathbf{0}_4$ and use it to show that

$$[H, \beta\boldsymbol{\Sigma}] = -2i\hbar c \begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix} \beta\boldsymbol{\nabla} \ .$$

Hint: Remember that the anticommutation relation between the Pauli matrices read $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbf{1}_2$. *(2 points)*

c) Using the commutators evaluated in a) and b), show that

$$[H, \beta\boldsymbol{\Sigma}] \cdot \boldsymbol{J} = \frac{1}{2}[H, \beta] \ ,$$

and conclude that the quantity

$$K = \beta\boldsymbol{\Sigma} \cdot \boldsymbol{J} - \frac{1}{2}\beta$$

commutes with H and is a constant of motion.

Hint: The angular momentum is a conserved quantity, $[H, \boldsymbol{J}] = 0$. *(3 points)*

Exercise 12 *Minimal coupling to the electromagnetic field*

Let us consider a particle of mass m and electric charge q coupled to an electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ via the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right),$$

with $\boldsymbol{v} = \dot{\boldsymbol{x}}$ being the velocity of the particle at position $\boldsymbol{x}$ at time t.

a) With the help of the relations $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, show that the Lorentz force can be written in terms of a potential U, such that

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i}$$

and determine explicitly the form of the potential U.

Hint: Use that $\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A}$. *(2 points)*

b) Determine the form of the Lagrangian L of the particle. *(1 point)*

c) What is the canonically-conjugated momentum $\boldsymbol{p}$ to the position $\boldsymbol{x}$? Derive then the form of the Hamiltonian H and determine the corresponding Hamilton equations. Show that they are equivalent to the Newton equation of motion. *(3 points)*