

Exercises for Theoretical Physics V

SoSe 2026

Sheet 7

22.05.2026

Exercise 13 *Hydrogen atom radial equation*

Consider the eigenstate problem of the Dirac equation for the hydrogen atom:

$$E\psi = H\psi,$$

where the Hamiltonian H is given by

$$H = -i\hbar c \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + mc^2 \beta - \frac{e^2}{r} \mathbf{1}_4. \quad (1)$$

with

$$\alpha^i = \begin{pmatrix} \mathbf{0}_2 & \sigma^i \\ \sigma^i & \mathbf{0}_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The total angular momentum is

$$\mathbf{J} = \frac{\mathbf{L}}{\hbar} \mathbf{1}_4 + \frac{\boldsymbol{\Sigma}}{2}$$

where $\boldsymbol{\Sigma} = (\Sigma^1, \Sigma^2, \Sigma^3)$, with $\Sigma^i = \begin{pmatrix} \sigma^i & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma^i \end{pmatrix}$, and the orbital angular momentum is

$$\mathbf{L} = -i\hbar \mathbf{r} \times \boldsymbol{\nabla}.$$

Both \mathbf{J} and the following quantity

$$K = \beta \boldsymbol{\Sigma} \cdot \mathbf{J} - \frac{1}{2} \beta \quad (2)$$

are conserved: $[H, J^i] = [H, K] = \mathbf{0}_4$.

a) Show that

$$\left(\frac{\boldsymbol{\Sigma} \cdot \mathbf{L}}{\hbar} \right)^2 = \frac{L^2}{\hbar^2} \mathbf{1}_4 - \frac{\boldsymbol{\Sigma} \cdot \mathbf{L}}{\hbar}$$

and hence that

$$K^2 = J^2 + \frac{\mathbf{1}_4}{4}.$$

Express the eigenvalues k of K in terms of j , where $j(j+1)$ are the eigenvalues of J^2 .

Hint: The Pauli matrices multiply as $\sigma^i \sigma^j = \delta^{ij} \mathbf{1}_2 + i \epsilon_{ijk} \sigma^k$, where ϵ_{ijk} is the Levi-Civita fully antisymmetric tensor and a summation over k is implied.

(3 points)

b) Show that

$$(\boldsymbol{\Sigma} \cdot \mathbf{r})(\boldsymbol{\Sigma} \cdot (\mathbf{r} \times \boldsymbol{\nabla})) = i(\boldsymbol{\Sigma} \cdot \mathbf{r})(\mathbf{r} \cdot \boldsymbol{\nabla}) - ir^2(\boldsymbol{\Sigma} \cdot \boldsymbol{\nabla})$$

and hence, recalling that $\mathbf{r} \cdot \boldsymbol{\nabla} = r(\partial/\partial r)$, show that

$$-ir^2\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} = (\boldsymbol{\alpha} \cdot \mathbf{r})(\boldsymbol{\Sigma} \cdot (\mathbf{r} \times \boldsymbol{\nabla})) - i(\boldsymbol{\alpha} \cdot \mathbf{r}) \left(r \frac{\partial}{\partial r} \right). \quad (3)$$

(2 points)

c) Let $\alpha_r = \frac{1}{r}\boldsymbol{\alpha} \cdot \mathbf{r}$ and, using Eqs. (2), (3) and the eigenvalues k of K , show that the hamiltonian (1) can be expressed in its radial form as

$$H = mc^2\beta - \frac{e^2}{r}\mathbf{1}_4 + i\hbar c\alpha_r \left(\frac{\beta k}{r} - \frac{1}{r}\mathbf{1}_4 - \mathbf{1}_4 \frac{\partial}{\partial r} \right). \quad (4)$$

(1 point)

d) Using the following two-component representation

$$\beta = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{1}_2 \end{pmatrix}, \quad \alpha_r = \begin{pmatrix} \mathbf{0}_2 & i\mathbf{1}_2 \\ -i\mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} u\mathbf{1}_2 \\ v\mathbf{1}_2 \end{pmatrix}$$

and the following definitions

$$a_1 = \frac{mc^2 - E}{\hbar c}, \quad a_2 = \frac{mc^2 + E}{\hbar c}, \quad \alpha = \frac{e^2}{\hbar c}, \quad (5)$$

show that the individual components of the stationary Dirac Equation $H\psi = E\psi$, with H given in (4), read

$$\begin{aligned} \left(\frac{\alpha}{r} - a_1 \right) u &= \left(\frac{k+1}{r} + \frac{\partial}{\partial r} \right) v \\ \left(\frac{\alpha}{r} + a_2 \right) v &= \left(\frac{k-1}{r} - \frac{\partial}{\partial r} \right) u. \end{aligned} \quad (6)$$

(1 point)

e) Defining $a = \sqrt{a_1 a_2}$ and functions f and g such that

$$u = \frac{e^{-ar}}{r} f, \quad v = \frac{e^{-ar}}{r} g,$$

show that Eq. (6) becomes

$$\begin{aligned} \left(\frac{\alpha}{r} - a_1 \right) f &= \left(\frac{k}{r} - a + \frac{\partial}{\partial r} \right) g \\ \left(\frac{\alpha}{r} + a_2 \right) g &= \left(\frac{k}{r} + a - \frac{\partial}{\partial r} \right) f. \end{aligned} \quad (7)$$

(1 point)

f) Assume a power series form of the functions

$$f = \sum_s c_s r^s, \quad g = \sum_s d_s r^s$$

where the exponents can be non-integer, but they differ sequentially by unity, that is: $\dots, s-1, s, s+1, s+2, \dots$. Show that the coefficients of Eq. (7) for r^{s-1} read

$$\begin{aligned} \alpha c_s - a_1 c_{s-1} &= (k+s)d_s - a d_{s-1} \\ \alpha d_s + a_2 d_{s-1} &= (k-s)c_s + a c_{s-1}. \end{aligned} \quad (8)$$

(1 point)

g) Impose that the wavefunction must be squared-integrable ($\int |\psi|^2 r^2 dr < \infty$) at $r = 0$: show that the power series coefficients c_s and d_s must vanish for $s < \epsilon$ and show that the minimal exponent ϵ must satisfy $\epsilon \geq -\frac{1}{2}$. Apply this constraint to Eq. (8) for $s = \epsilon$ and show that

$$\alpha^2 + \epsilon^2 - k^2 = 0. \quad (9)$$

Explain why the solution of Eq. (9) with negative ϵ is not acceptable.

(3 points)

h) From Eq. (8) derive the following relation

$$[\alpha a + a_1(k-s)]c_s = [\alpha a_1 + a(k+s)]d_s. \quad (10)$$

Approximate Eq. (8) and (10) for large s and show that

$$\frac{c_s}{c_{s-1}} \approx \frac{2a}{s}.$$

Show that this implies that $f \approx e^{2ar}$ for large r .

(3 points)

i) If a is real, the power series coefficients must vanish for $s > \epsilon + n$ with $n \geq 1$, so that the exponential does not explode at infinity. Apply this constraint to Eq. (8) and derive the relation

$$(a_2 - a_1)\alpha = 2a(\epsilon + n). \quad (11)$$

Use Eqs. (11), (9) and (5) to show that

$$E = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(n + \sqrt{k^2 - \alpha^2})^2}}}.$$

(2 points)