Exercises for
Theoretical Physics V

SoSe 2026 **Sheet 8** **27.05.2026**

Exercise 14 *Gauge transformation and the wave function*

We consider a single electron, $q = -e$, in the non-relativistic limit in a region of vanishing magnetic field $\boldsymbol{B} = 0$. The minimal-coupling Hamiltonian for the gauge choice $\phi = 0$ reads

$$H = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 . \tag{1}$$

a) Let $\psi^{(0)}(\boldsymbol{r}, t)$ be the solution of the Schrödinger equation for the case of a vanishing vector potential ($\boldsymbol{A} = 0$):

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi^{(0)} = -\frac{\hbar^2\nabla^2}{2m}\psi^{(0)} . \tag{2}$$

Verify that $\psi(\boldsymbol{r}, t) = \mathcal{S}(\boldsymbol{r})\psi^{(0)}(\boldsymbol{r}, t)$, with

$$\mathcal{S}(\boldsymbol{r}) = \exp\left(-\frac{\mathrm{i}e}{\hbar c}\int^{s(\boldsymbol{r})} \mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{A}(\boldsymbol{r}')\right) , \tag{3}$$

satisfies the Schrödinger equation for a non-vanishing $\boldsymbol{A}$. Here, $s(\boldsymbol{r})$ denotes a generic path with endpoint $\boldsymbol{r}$. *(2 points)*

b) We now perform a gauge transformation: $\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda$, with some scalar function $\Lambda(\boldsymbol{r})$. Show that the solution ψ' of the Schrödinger equation for $\boldsymbol{A}'$ is related to the previous solution for $\boldsymbol{A}$ via $\psi' = \mathcal{U}\psi$. Determine the explicit form of the transformation $\mathcal{U}$. *(1 point)*

Exercise 15 *Aharonov-Bohm effect*

Consider an electron (charge $q = -e$) moving in a region where the magnetic field $\boldsymbol{B} = 0$ everywhere except inside an infinitely long solenoid. The vector potential $\boldsymbol{A}$ outside the solenoid is non-zero, and the magnetic flux through the solenoid is Φ. An electron beam is split into two parts that travel along two different paths, Path 1 and Path 2, which enclose the solenoid and recombine at a point $\boldsymbol{r}$.

a) Write the wavefunction

$$\psi(\boldsymbol{r}) = \psi_1(\boldsymbol{r}) + \psi_2(\boldsymbol{r})$$

at the recombination point in terms of the free wavefunctions of the two paths $\psi_1^{(0)}(\boldsymbol{r})$ and $\psi_2(\boldsymbol{r})$ (which both satisfy the Schrödinger equation with $\boldsymbol{A} = 0$) and the vector potential $\boldsymbol{A}$.

(1 point)

b) Derive the expression for the phase difference between the two paths at $\boldsymbol{r}$ and show that it depends on the magnetic flux Φ enclosed by the two paths.

Hint: Recall the Stokes theorem $\oint_\gamma d\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}) = \int_{S_\gamma} dS(\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{n}$, where $\oint_\gamma$ is a line integral over a closed curve γ, $\int_{S_\gamma}$ is a surface integral over a the surface S_γ enclosed by γ, and $\boldsymbol{n}$ is the normal vector.

(1 point)