Exercises for Theoretical Physics V

SoSe 2026 **Sheet 10** **09.06.2026**

Exercise 18 *The Quantum String*

Consider the Hamiltonian

$$\hat{H} = \sum_{j=1}^{N} \left(\frac{\hat{p}_j^2}{2m} + \frac{\kappa}{2} (\hat{q}_j - \hat{q}_{j+1})^2 \right) , \tag{1}$$

where the positions $\hat{q}_j$ and momenta $\hat{p}_j$ are canonically-conjugated variables, satisfying the commutation relations

$$[\hat{p}_j, \hat{q}_k] = -i\hbar\delta_{jk} \, . \tag{2}$$

We impose here periodic boundary conditions, such that $\hat{q}_j = \hat{q}_{j+N}$ and $\hat{p}_j = \hat{p}_{j+N}$. In this exercise, our objective is to show that $\hat{H}$ can be written as the sum of harmonic oscillators.

a) We start by expanding the operators $\hat{q}_j, \hat{p}_j$ in Fourier series:

$$\hat{q}_j(t) = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} \hat{Q}_n(t) e^{\mathrm{i}2\pi nj/N} \quad \text{and} \quad \hat{p}_j(t) = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} \hat{P}_n(t) e^{\mathrm{i}2\pi nj/N} \, . \tag{3}$$

Verify that the series in Eq. (3) obey the periodic boundary conditions $\hat{q}_j = \hat{q}_{j+N}$ and $\hat{p}_j = \hat{p}_{j+N}$. Given the hermiticity of $\hat{q}_j$ and $\hat{p}_j$ and the commutation relation (2), show that the operators $\hat{Q}_n$ and $\hat{P}_n$ shall fulfill the relations

$$[\hat{P}_n^\dagger, \hat{Q}_m] = -i\hbar\delta_{nm} \, , \quad \hat{P}_n^\dagger = \hat{P}_{-n} \, , \quad \hat{Q}_n^\dagger = \hat{Q}_{-n} \, . \tag{4}$$

(2 points)

b) Show that using Eqs. (3) and (4), the Hamiltonian (1) can be recast in the form

$$\hat{H} = \sum_{n=-N/2}^{N/2} \left(\frac{\hat{P}_n^\dagger \hat{P}_n}{2m} + \frac{1}{2} m\omega_n^2 \hat{Q}_n^\dagger \hat{Q}_n \right) , \tag{5}$$

with

$$\omega_n^2 = \frac{4\kappa}{m} \sin^2\left(\frac{\pi n}{N}\right) . \tag{6}$$

(2 points)

c) We further define the operators $\hat{a}_n, \hat{a}_n^\dagger$, such that

$$\hat{Q}_n = \sqrt{\frac{\hbar}{2m\omega_n}} (\hat{a}_{-n}^\dagger + \hat{a}_n) \quad \text{and} \quad \hat{P}_n = \mathrm{i}\sqrt{\frac{\hbar m\omega_n}{2}} (\hat{a}_{-n}^\dagger - \hat{a}_n) \, . \tag{7}$$

Derive the commutation relations for $\hat{a}_n$ and $\hat{a}_n^\dagger$. Subsequently, show that the Hamiltonian (5) can be rewritten as

$$\hat{H} = \sum_{n=-N/2}^{N/2} \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right) . \tag{8}$$

What are the eigenvalues of the Hamiltonian? Discuss why ω_n in Eq. (8) is positive.

(2 points)

d) Consider the ground-state energy of Eq. (8) and determine its expression in the continuum limit by taking $N \to \infty$. *(1 point)*

Exercise 19 *Photons with circular polarization*

Let us consider a monochromatic electromagnetic field with a wavevector $\boldsymbol{k}$ and two possible linear polarisations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, such that it respect the conditions $\boldsymbol{k} \cdot \boldsymbol{e}_1 = \boldsymbol{k} \cdot \boldsymbol{e}_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$. The vectors $\boldsymbol{e}_{1,2}$ are unit vectors.
The energy of the quantized electromagnetic field reads

$$\hat{H} = \hbar c |\boldsymbol{k}| (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1), \tag{9}$$

where the operators $\hat{a}_j^\dagger$ act on the vacuum to create a photon at wavelength $\boldsymbol{k}$ and polarization $\boldsymbol{e}_j$. The annhihilation and creation operators obey bosonic commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{10}$$

$$[\hat{a}_i^{(\dagger)}, \hat{a}_j^{(\dagger)}] = 0. \tag{11}$$

a) We shall consider the circular polarization vectors

$$\boldsymbol{e}_\pm = \mp \frac{1}{\sqrt{2}} (\boldsymbol{e}_1 \pm i \boldsymbol{e}_2). \tag{12}$$

Using the definition of the quantized photon field

$$\hat{\boldsymbol{A}}(\boldsymbol{r}, t) = \sqrt{\frac{2\pi c^2 \hbar}{\omega V}} \sum_j \left[\hat{a}_j(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_j + \hat{a}_j^\dagger(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_j^* \right] , \tag{13}$$

show that it can written in terms of operators $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$ annihilating and creating photons with polarization $\boldsymbol{e}_\pm$. Determine the explicit form of $\hat{a}_\pm$ and show that they obey to bosonic commutation relations.

(2 points)

b) Let us apply to $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ a rotation around the wavevector $\boldsymbol{k}$ by an infinitesimal angle $\delta\theta$. Show that the polarization vectors are changed by a factor $\delta \boldsymbol{e}_\pm = \mp i \delta\theta \boldsymbol{e}_\pm$. What can you deduce about the spin of the photon ?

Hint: We remind that for a two-level system, the rotation operator $\hat{R}(\theta)$ by an angle θ around the axis $\boldsymbol{n}$ reads

$$\hat{R}(\theta) = \exp(i\theta(\boldsymbol{n} \cdot \boldsymbol{\sigma})) = I_2 \cos\theta + i(\boldsymbol{n} \cdot \boldsymbol{\sigma}) \sin\theta,$$

where the elements of the vector $\boldsymbol{\sigma}$ are Pauli matrices.

(2 points)