Quantum Optics

WS 2017/18 **Sheet 2** **03.11.2017**

Exercise 1

a) In a two-level system the transition between the ground state $|g\rangle$ and the excited state $|e\rangle$ has the transition frequency ω_0.

Exactly solve the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_t = \hat{H}|\psi\rangle_t \tag{1}$$

where the Hamiltonian read

$$\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z + \hbar\Omega(\hat{\sigma}^+ + \hat{\sigma}^-) \tag{2}$$

and with initial conditions $|\psi\rangle_0 = \alpha(0)|g\rangle + \beta(0)|e\rangle$. Here $\Delta = \omega - \omega_0, \Omega$ are real numbers, $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$ and $\hat{\sigma}^- = |g\rangle\langle e|$, $\hat{\sigma}^+ = (\hat{\sigma}^-)^\dagger$. *(2 Points)*

b) Solve the Schrödinger equation using perturbation theory to first order when $|\psi\rangle_0 = |g\rangle$ assuming Ω/Δ is the small parameter. *(2 Points)*

Exercise 2

Assume that a third level $|i\rangle$ at frequency $\omega_1 > \omega_0$ can be coupled to state $|g\rangle$ via radiation. The Hamiltonian is given by

$$\begin{aligned}\hat{H} =& \hbar\omega_0|e\rangle\langle e| + \hbar\omega_1|i\rangle\langle i| \\ &+ \hbar\Omega(|e\rangle\langle g|e^{-i\omega t} + |g\rangle\langle e|e^{i\omega t}) + \hbar\Omega'(|i\rangle\langle g|e^{-i\omega t} + |g\rangle\langle i|e^{i\omega t}).\end{aligned} \tag{3}$$

a) Find the representation in which the Hamiltonian is time independent. *(2 Points)*

b) Determine the condition under which the coupling to level $|i\rangle$ can be neglected and the system can be reduced to two levels. *(2 Points)*

Exercise 3

The dynamics of the density matrix $\hat{\rho}$ is governed by the master equation

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}] + \frac{\Gamma}{2}(2\hat{\sigma}^-\hat{\rho}\hat{\sigma}^+ - \hat{\sigma}^+\hat{\sigma}^-\hat{\rho} - \hat{\rho}\hat{\sigma}^+\hat{\sigma}^-) \tag{4}$$

where $\Gamma > 0$,

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega(\hat{\sigma}^+e^{-i\omega t} + \hat{\sigma}^-e^{i\omega t}) \tag{5}$$

and with same notations as in Exercise 1.

a) Determine the form of the master equation when $\hat{H}$ is moved to the reference frame which is time independent. *(2 Points)*

b) Write the optical Bloch equations. *(1 Point)*

c) Solve the optical Bloch equations for the initial condition

$$\hat{\rho}_{t=0} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\Omega = 0$. Determine $\mathrm{Tr}(\hat{\rho}^2)$ as a function of time. *(1 Point)*

d) Solve the optical Bloch equations for $\Gamma = 0$, $\Omega > 0$ and $\Delta = \omega - \omega_0 > 0$ with initial condition

$$\hat{\rho}_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Determine $\mathrm{Tr}(\hat{\rho}^2)$ as a function of time. *(1 Point)*

Exercise 4

Consider the matrix $\hat{\rho} = (\hat{1} + \mathbf{U} \cdot \hat{\boldsymbol{\sigma}})/2$ with

$$\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)^T \tag{6}$$

and

$$\mathbf{U} = (U_x, U_y, U_z)^T \in \mathbb{R}^3 \tag{7}$$

where $\hat{1}$ is the unity matrix and $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are the Pauli matrices.

a) Show that $\hat{\rho}$ is a density matrix if and only if $|\mathbf{U}| \leq 1$. *(2 Points)*

b) For which $\mathbf{U} \in \mathbb{R}^3$ the matrix $\hat{\rho}$ is a pure state? *(1 Point)*