Quantum Optics

WS 2017/18 **Sheet 4** **22.12.2017**

Exercise 7

Given an harmonic oscillator with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ such that $[\hat{a}, \hat{a}^\dagger] = 1$, let $\alpha \in \mathbb{C}$ and $|\alpha\rangle$ the coherent state

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \sum_n c_n |n\rangle. \tag{1}$$

Here $|n\rangle$ are the Fock states fulfilling $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

a) Derive the coefficients c_n and show

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\,. \tag{2}$$

(1 Point)

b) The position $\hat{x}$ and momentum $\hat{p}$ operators are defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^\dagger + \hat{a})\,, \tag{3}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^\dagger - \hat{a})\,. \tag{4}$$

Show that

$$\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}\,, \tag{5}$$

$$\Delta\hat{p} = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}}\,, \tag{6}$$

where the expectation value of an observable $\hat{O}$ is taken over a coherent state $\langle\hat{O}\rangle = \langle\alpha|\hat{O}|\alpha\rangle$. What does that mean for $\Delta\hat{x}\Delta\hat{p}$. *(2 Points)*

c) Derive the explicit form of $\langle x|\alpha\rangle$. Use that

$$\langle x|\hat{a}|\alpha\rangle = \alpha\langle x|\alpha\rangle \tag{7}$$

and

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + i\frac{\hat{p}}{m\omega}\right). \tag{8}$$

Find a differential equation for $\langle x|\alpha\rangle$ and solve it. *(2 Points)*

Exercise 8

With the same notations as in the previous exercise we define the squeezed state

$$|\xi\rangle = \hat{S}(\xi)|0\rangle = \sum_{n=0}^{\infty} d_n|n\rangle. \tag{9}$$

Here

$$\hat{S}(\xi) = \exp\left(\frac{\xi^*\hat{a}^2 - \xi(\hat{a}^\dagger)^2}{2}\right) \tag{10}$$

is the squeezing operator. In what follows we use $\xi = re^{-i2\phi}$.

a) Derive the relations

$$\hat{S}(\xi)\hat{a}\hat{S}(\xi)^\dagger = \hat{a}\cosh(r) + e^{-i2\phi}\hat{a}^\dagger\sinh(r)\,, \tag{11}$$

$$\hat{S}(\xi)\hat{a}^\dagger\hat{S}(\xi)^\dagger = \hat{a}^\dagger\cosh(r) + e^{i2\phi}\hat{a}\sinh(r)\,. \tag{12}$$

and show that

$$(\hat{a}\cosh(r) + e^{-i2\phi}\hat{a}^\dagger\sinh(r))|\xi\rangle = 0. \tag{13}$$

Show furthermore that

$$\hat{S}(\xi)^\dagger\hat{a}\hat{S}(\xi) = \hat{a}\cosh(r) - e^{-i2\phi}\hat{a}^\dagger\sinh(r)\,, \tag{14}$$

$$\hat{S}(\xi)^\dagger\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh(r) - e^{i2\phi}\hat{a}\sinh(r)\,. \tag{15}$$

(2 Points)

b) Use the relation in Eq. (13) to calculate the coefficients d_n and show

$$d_{2n+1} = \langle 2n+1|\xi\rangle = 0,$$

$$d_{2n} = \langle 2n|\xi\rangle = (-1)^n\sqrt{\frac{(2n-1)!!}{(2n)!!}}e^{-in2\phi}\tanh^n(r)d_0.$$

Here we used the definition $(n)!! = n(n-2)(n-4)\times\ldots\times 1$. *(2 Points)*

c) Determine d_0 from the equation $\sum_{n=0}^{\infty}|d_n|^2 = 1$. Use that

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = (1-4x)^{-1/2}.$$

(1 Point)

d) Determine the variance $\Delta\hat{x}$ and $\Delta\hat{p}$ with the definitions of Eqs. (3), (4) and verify

$$(\Delta\hat{x})^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2 = \frac{\hbar}{2m\omega}\left(\frac{1-\cos(2\phi)}{2}e^{2r} + \frac{1+\cos(2\phi)}{2}e^{-2r}\right)\,, \tag{16}$$

$$(\Delta\hat{p})^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = \frac{\hbar m\omega}{2}\left(\frac{1+\cos(2\phi)}{2}e^{2r} + \frac{1-\cos(2\phi)}{2}e^{-2r}\right)\,, \tag{17}$$

where the expectation values are now evaluated with state $|\xi\rangle$. *(1 Point)*

e) Is the state for $\phi = \pi/4$ squeezed? Define two new operators

$$\hat{x}_\theta = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger e^{-i\theta} + \hat{a}e^{i\theta}), \tag{18}$$

$$\hat{p}_\theta = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger e^{-i\theta} - \hat{a}e^{i\theta}) \tag{19}$$

and calculate the variances $\Delta\hat{x}_\theta^2$ and $\Delta\hat{p}_\theta^2$. For what choice of θ the product $\Delta\hat{x}_\theta\Delta\hat{p}_\theta$ is minimal.

(2 Points)