Quantum Optics

WS 2017/18 **Sheet 5** **19.01.2017**

Exercise 9

Consider the interaction between the two-level atom and the cavity field described by the Jaynes-Cummings Hamiltonian:

$$\hat{H} = \hbar\omega_0 \hat{\sigma}^\dagger \hat{\sigma} + \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar g(\hat{a}^\dagger \hat{\sigma} + \hat{\sigma}^\dagger \hat{a}), \tag{1}$$

where the coupling constant g is real, ω_0 is the atomic transition frequency and ω_c is the cavity resonance frequency. The rising and lowering operators for the atomic internal state is given by $\hat{\sigma}^\dagger = |e\rangle\langle g|$, $\hat{\sigma} = |g\rangle\langle e|$ where $|g\rangle$ and $|e\rangle$ denote the atomic ground and excited state. The operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation an creation operators for the cavity mode.

a) Calculate the spectrum and the eigenvectors of the Hamiltonian $\hat{H}$. Discuss the results as functions of photon number n and of the detuning $\delta = \omega_c - \omega_0$. *(2 Points)*

b) Solve the Schrödinger equation

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \tag{2}$$

for the initial condition $|\psi\rangle_0 = |g\rangle \sum_{n=0}^{\infty} c_n |n\rangle$ with $\sum_{n=0}^{\infty} |c_n|^2 = 1$. *(1 Point)*

Exercise 10

Consider an optical resonator crossed by a beam of atoms. The light-atom interaction time τ is much smaller than the inverse injection rate $1/r$ and the inverse coupling strength $(g\sqrt{\langle \hat{n}\rangle})^{-1}$, where $\langle \hat{n}\rangle = \langle \hat{a}^\dagger \hat{a}\rangle$ is the mean photon number. Therefore the following inequalities are fulfilled: $r\tau \ll 1$ and $g\sqrt{\langle \hat{n}\rangle}\tau \ll 1$. In this case the field evolution can be described by the following master equation:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{\gamma_g}{2}(\hat{a}^\dagger \hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger \hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger) - \frac{\gamma_e}{2}(\hat{a}\hat{a}^\dagger \hat{\rho} + \hat{\rho}\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger \hat{\rho}\hat{a}), \tag{3}$$

where the oscillator damping and the pumping rates $\gamma_g = r_g g^2 \tau^2$ and $\gamma_e = r_e g^2 \tau^2$ are proportional to the injection rate of the atoms in the ground sate r_g or in the excited state r_e, respectively.

a) For $\gamma_e > \gamma_g$ find a time scale t till which the dynamic evaluated from Equation (3) is valid, given that the cavity is initially prepared in the vacuum state with density matrix $\hat{\rho}(0) = |0\rangle\langle 0|$. *(1 Point)*

b) For $\gamma_g > \gamma_e$ show that the steady state is diagonal in the Fock basis representation. Derive the steady state. *(2 Points)*

c) Derive a master equation from the Jaynes-Cummings Hamiltonian in Eq. (1) under the assumption that the atoms are injected with rate r and state $|\psi\rangle_{\text{at}} = \alpha|g\rangle + \beta|e\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. *(3 Points)*