## 8. Binomial distribution

We consider $N$ independent coin tosses with a flipped coin. Heads occur with probability $p$, tails with probability $1-p$. The probability of finding $n$ times heads after $N$ coin tosses is given by the binomial distribution

$$W_N(n) = \binom{N}{n} p^n (1-p)^{N-n} \tag{1}$$

(a) Is $W_N(n)$ normalized ?

(1 Point)

(b) Compute $\langle n \rangle$ and $\langle n^2 \rangle$.

(1 Point)

(c) Show that the $k$th moment is given by the following formula:

$$\langle n^k \rangle = \left( p \frac{\partial}{\partial p} \right)^k (p+q)^N \Bigg|_{q=1-p} \tag{2}$$

(1 Point)

(d) Calculate the variance $\mathrm{Var}(n) = \langle (n - \langle n \rangle)^2 \rangle$. How does $\sqrt{\mathrm{Var}(n)}/\langle n \rangle$ scale for size $N$?

(1 Point)

## 9. Poisson distribution und Stirling formula

As in exercise 8, we consider the binomial distribution. Here we focus on the limiting case of small probabilities of occurrence $p \ll 1$ (condition A) and large numbers of trials $n \ll N$ (condition B). We now want to find an approximation of the binomial distributions in this limiting case. Proceed as follows:

(a) Derive the Stirling formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \text{ für } n \gg 1 \tag{3}$$

as the approximation of the factorial for large numbers $n$. Use the gamma function as a continuous continuation of the factorial.

$$n! = \Gamma(n+1) = \int_0^{+\infty} \mathrm{d}x e^{n\ln(x)-x} \tag{4}$$

and substitute $y = x/n$. Approximate the remaining integral using the saddle point approximation

$$\int_a^b \mathrm{d}x e^{nf(x)} \approx \sqrt{\frac{2\pi}{n|f''(x_0)|}} e^{nf(x_0)}, \text{ für } n \gg 1, \tag{5}$$

for $f$ twice continuously differentiable, arbitrary endpoints $a < b$, and $x_0$ the global maximum of $f$.

(1 Point)

(b) Show that $(1-p)^{N-n} \approx e^{-Np}$.

(1 Point)

(c) Show that $N!/(N-n)! \approx N^n$

(1 Point)

(d) Using the previous part of the exercise, derive the Poisson distribution

$$W_N(n) = \frac{\lambda^n e^{-\lambda}}{n!} \tag{6}$$

as a limiting case of the binomial distribution under conditions A and B, i.e. for a large number of trials and small probabilities of occurrence $p$, where $\lambda = Np$. Calculate the expectation value and variance of the distribution.

(1 Point)

(e) In a game of roulette, a ball is thrown in each game and falls randomly on a square with numbers 0 to 36, with each number occurring only once. A coup is a series of 37 games. Calculate the average number of different numbers that are hit during a coup. Do this once using the binomial distribution and once using the Poisson distribution and compare the results obtained.

(1 Point)

**10. Properties of paramagnetic systems**

Consider a system consisting of a set of $N$ spins $\frac{1}{2}$ in the presence of a $z$-oriented magnetic field $\vec{B} = B\vec{e}_z$. The Hamiltonian describing the system is then

$$\hat{H} = -\mu_B B \sum_i \hat{\sigma}_i^z, \tag{7}$$

where $\mu_B$ is the Bohr magneton and $\hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli spin operator for the $i$-th spin. Let $p(\sigma_1, \ldots, \sigma_N$ be the probability that the system is in the microstate $|\sigma_1, \ldots, \sigma_N\rangle$:

(a) Determine the reduced probability $p_1(\sigma_i)$ that the $i$-th spin is in the configuration $\sigma_i$, depending on $p(\sigma_1, \ldots, \sigma_N)$.

(1 Point)

(b) Derive from the expression of the reduced probability the mean value $\langle \hat{M}^z \rangle$ of the total magnetisation, defined as $\hat{M}^z = -\mu_B \sum_i \hat{\sigma}_i^z$.

(1 Point)

(c) Determine the expression for the variance of the magnetisation

$$\Delta M^z = \sqrt{\langle (\hat{M}^z)^2 \rangle - \langle \hat{M}^z \rangle^2} \tag{8}$$

and show that it can be represented in terms of the reduced probability $p_1(\sigma_i)$ and the joint probability $p_2(\sigma_i, \sigma_j)$.

(1 Point)

(d) What is the mean value of the magnetisation $\langle \hat{M}^z \rangle$ in the case where the probability of finding the system in the configuration $|\sigma_1, \ldots, \sigma_N\rangle$ is uniformly distributed $p(\sigma_1, \ldots, \sigma_N) = 1/2^N$? How large is then the variance $\Delta M^z$ and how does it scale with the system size $N$?

(1 Point)