16. Entropic elasticity

For a simple model of entropic elasticity, consider a one-dimensional chain consisting of N elements. Each element of the chain has the length a and should be able to align itself either parallel or antiparallel to the connection of the two end points. n elements of the chain are aligned antiparallel to the connection of the two end points, the remaining $N - n$ elements are aligned parallel to it. Let the approximation $N \gg 1$, $n \gg 1$ and $N - n \gg 1$ be valid.

(a) Calculate the number of microstates belonging to the state described above. Give a relationship between N, n and the length x of the chain. (1 point)

(b) How does the entropy of the system depend on the length of the chain? Show that the entropy decreases with the length of the chain, i.e. $\partial_x S < 0$. What does this mean for the chain, assuming that the system tries to adopt a configuration where entropy is maximal.

(2 points)

17. Equivalent definitions of entropy

We consider a microcanonical ensemble with N identical particles, a volume V and an energy that takes values between $E - \Delta$ and E ($\Delta > 0$). Show that the following three formulae lead to an equivalent definition of the entropy S, differing only by an additive constant of order $\mathcal{O}(\ln N)$.

$$S = k_B \ln \Gamma(E) \tag{1a}$$

$$S = k_B \ln \omega(E) \tag{1b}$$

$$S = k_B \ln \Sigma(E). \tag{1c}$$

Here $\Gamma(E)$ is the phase space volume occupied by the totality of all microstates with

$$\Gamma(E) = \int_{E-\Delta}^{E} \mathrm{d}E' \mathrm{Tr}\{\delta(E' - \hat{H})\}, \tag{2}$$

$\Sigma(E)$ is the phase space volume bounded by the area of energy E with

$$\Sigma(E) = \mathrm{Tr}\{\theta(E - \hat{H})\}, \tag{3}$$

and $\omega(E)$ is the density of states of the system at energy E which is defined as

$$\omega(E) = \frac{\partial \Sigma(E)}{\partial E}. \tag{4}$$

Hint: You may assume that $\Delta \ll E$ and that the hyperbody described by the integrand in Eq.(2) is such that $\Sigma(E-\Delta)/\Sigma(E) \to 0$, for fixed Δ and $N \to \infty$.

(2 points)

18. Microcanonical description of paramagnets

Consider again a system of N distinct spin-1/2 particles in a magnetic field. The individual spins can only assume two energy values $E_- = -E_0/2$ or $E_+ = +E_0/2$. Let n_- be the occupation of E_- and n_+ the occupation of E_+. Let the total energy of the system be U.

(a) Determine the entropy of the system in the microcanonical ensemble.

(1 point)

(b) Find the most probable values for n_+ and determine the standard deviation from these values.

(1 point)

(c) Determine the temperature and show that it can take negative values.

(1 point)

(d) Suppose that a negative temperature reservoir is brought into contact with a positive temperature reservoir. In which direction does the heat flow?

(1 point)

19. The von Neumann equation

Show that the formal solution of the von Neumann equation

$$\partial_t \rho_t = \frac{1}{i\hbar}[\hat{H}, \rho_t] \tag{5}$$

is given by the following equation

$$\hat{\rho} = e^{-i\hat{H}t/\hbar}\hat{\rho}_0 e^{i\hat{H}t/\hbar}, \tag{6}$$

where $\hat{\rho}_0$ is the density matrix at time $t = 0$.

(1 point)