21. Fermi gas piston

In this problem we consider a cylinder whose volume is divided into two subvolumes 1 and 2 by a freely movable piston. There are Fermi gases in both subvolumes. The particles in volume 1 have a spin of $1/2$ and the particles in volume 2 have a spin of $3/2$. Both gases have the same temperature T. Determine the ratio of the densities of the two gases at which the system is in equilibrium in the limits $T \to 0$ and $T \to \infty$.

(3 points)

22. Relativistic Fermi gas and white dwarfs

In this exercise, we deal with the pressure and density profile inside a spherical, non-rotating white dwarf. For this purpose, we assume that the white dwarf consists entirely of ionised ^{4}He. The electron gas can be considered as completely degenerate, since its temperature $T \approx 10^6$K is much smaller than the typical Fermi temperature $T_F = 10^9$K. The relativistic electron gas counteracts the gravitational pressure generated by the helium nuclei, so that a hydrostatic equilibrium is established.

In order to describe the local equilibrium state of a small part of the electron gas inside the white dwarf, we consider the grandcanonical partition function for a relativistic Fermi gas, which is located in a box with the edge length L and the volume V (periodic boundary conditions). The volume is chosen so small that both the local electron density and the density of the helium nuclei can be regarded as constant. The electron spin is given by $s = 1/2$, the corresponding degeneracy factor is $g = 2s + 1 = 2$ and $n_e = \langle N_e \rangle / V$ is the local density of the electron gas. Then the logarithm of the grand canonical sum of states of the Fermi gas is

$$\ln \mathcal{Q}(V, \beta, z) = g \sum_{\boldsymbol{p}} \ln(1 + z e^{-\beta \epsilon_{\boldsymbol{p}}}), \text{ with } \epsilon_{\boldsymbol{p}} = m_e c^2 \sqrt{1 + \left(\frac{\boldsymbol{p}}{m_e c}\right)^2} \quad (1)$$

the relativistic one-particle energy of an electron with momentum $\boldsymbol{p}$ and mass $m_e = 9.11 \times 10^{-31}$kg.

(a) Approximate the sum of states for large $L \gg 1$ by an integral over all pulses and show from $\Omega = -k_B T \ln \mathcal{Q}(v, \beta, z) = -PV$ that the pressure of the electron gas P is given by the integral

$$P = \frac{4\pi g}{3h^3} \int_0^\infty \frac{\mathrm{d}\epsilon}{\mathrm{d}p} \frac{p^3 \mathrm{d}p}{z^{-1} e^{\beta\epsilon(p)} + 1}, \tag{2}$$

where $\epsilon(p) = \epsilon_{\boldsymbol{p}}$ with $p = |\boldsymbol{p}|$. Calculate the integral for the limit case $T \to 0$ by using the relation

$$\lim_{T \to 0} (z^{-1} e^{\beta\epsilon(p)} + 1)^{-1} = \theta(\epsilon(p_F) - \epsilon(p)) \tag{3}$$

The Fermi momentum p_F corresponds to the radius of the momentum sphere and is defined by the relation

$$g \frac{4\pi V}{h^3} \int_0^{p_F} \mathrm{d}p p^2 = \frac{gV}{h^3} \frac{4\pi p_F^3}{3} = \langle N_e \rangle \tag{4}$$

Using the substitution $p/m_e c = \sinh x$, show that the following expression for the electron pressure is obtained

$$P = \frac{\pi g m_e^4 c^5}{6h^3} f\left(\frac{p_F}{m_e c}\right), \text{ where } f(y) = y(2y^2 - 3)\sqrt{1 + y^2} + 3\sinh^{-1}(y). \tag{5}$$

(3 points)

(b) Determine the leading order of expression (5) for the non-relativistic limit case $p/m_e c \ll 1$ using the series expansion $\sinh^{-1}(y) = -y^3/6 + 3y^5/40 + \mathcal{O}(y^7)$. Use this to show that substituting the Fermi momentum yields the outward pressure,

$$\frac{p}{m_e c} \ll 1 \;:\; P = C_{\mathrm{nr}} n_e^{\gamma_{\mathrm{nr}}}, \text{ mit } C_{\mathrm{nr}} = \frac{g\hbar^2}{30\pi^2 m_e} \left(\frac{6\pi^2}{g}\right)^{5/3}, \text{ und } \gamma_{\mathrm{nr}} = \frac{5}{3} \tag{6}$$

where γ_{nr} denotes the adiabatic exponent in the non-relativistic case.

(1 point)

(c) Calculate the leading order of expression (5) for the ultrarelativistic case $p/m_e c \gg 1$. Use the symptotic development $\sinh^{-1}(y) = \ln(2y) + \mathcal{O}(y^{-2})$ for $y \gg 1$ and derive the following expression:

$$\frac{p}{m_e c} \gg 1 \;:\; P = C_{\mathrm{ur}} n_e^{\gamma_{\mathrm{ur}}}, \text{ mit } C_{\mathrm{ur}} = \frac{gc\hbar}{24\pi^2} \left(\frac{6\pi^2}{g}\right)^{4/3}, \text{ und } \gamma_{\mathrm{nr}} = \frac{4}{3}. \tag{7}$$

(1 point)

(d) We now come to the calculation of the spherically symmetric pressure and mass density profile $P(r)$ and $\rho(r)$ inside a white dwarf $(0 < r \neq R)$, whose radius and mass we denote by R and M. First, we note that the electron density $n_e(r) = \langle N_e \rangle / V$ depends on the distance r between the volume V and the centre of the white dwarf. Since, due to the complete ionisation, there are two electrons for every four nucleons of the helium nucleus, the mass density $\rho(r) = 2m_u n_e(r)$, with $m_u = 1.66 \times 10^{-27}\mathrm{kg}$. Since the mass of the proton and neutron is larger than that of the electron by a factor of 2000, the contribution of the electrons to the mass density can be neglected as a good approximation. In the following, we restrict ourselves to the cases of the non-relativistic and the ultra-relativistic electron gas investigated in problem parts b) and c). Using the expressions for the electron density from b) and c), we find the following relationship between pressure and mass density

$$P(r) = K(\rho(r))^{\gamma} = K(\rho(r))^{\frac{1+n}{n}} \tag{8}$$

with the constant $K \in \{C_{\mathrm{nr}}/(2m_u)^{\gamma_{\mathrm{nr}}}, C_{\mathrm{ur}}/(2m_u)^{\gamma_{\mathrm{ur}}}\}$, the adiabatic exponent $\gamma \in \{\gamma_{\mathrm{nr}}, \gamma_{\mathrm{ur}}\}$ and the corresponding polytropic index $n = \frac{1}{\gamma-1}$. Derive the equation for hydrostatistic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r) \tag{9}$$

and the equation for mass conservation

$$M(r) = \int_0^r \mathrm{d}u \rho(u) u^2 \tag{10}$$

as well as the following relation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho(r) \tag{11}$$

Here $M(r)$ is the mass inside the sphere with radius r and $G = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is the Newtonian gravitational constant. (1 point)

23. Heat capacity of the electron gas

Let us consider a gas of N free fermions of mass m_e, with spin $S = 1/2$. The purpose of this exercise is to provide a perturbative expansion of U, the internal energy of a Fermi gas, in the limit of low temperatures ($T \ll T_F$). We remind that in the limit of a system sufficiently large we can take the continuous limit of the sum

$$U = \sum_{\boldsymbol{p}} \epsilon_{\boldsymbol{p}} \langle n_{\boldsymbol{p}} \rangle = \frac{V}{(2\pi\hbar)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{p}\, \epsilon_{\boldsymbol{p}} n_{\boldsymbol{p}}, \tag{12}$$

with $\epsilon_{\boldsymbol{p}} = |\boldsymbol{p}|^2/2m_e$.

(a) Show that in the limit where the temperature is $T = 0$, the internal energy is expressed in terms of the Fermi energy ϵ_F and follows the expression

$$U = \frac{3}{5} N \epsilon_F. \tag{13}$$

Hint: You may use Eq. (4) which provides a relation between the Fermi impulsion and the number of particles N

(1 point)

(b) We now consider the case of a system at finite temperature T much smaller than the Fermi temperature $T \ll T_F$. Show that the internal energy can be rewritten such that

$$U = \frac{V}{4\pi^2\hbar^3 m} \int_0^\infty \mathrm{d}p \frac{p^5}{5} \left(-\frac{\partial}{\partial p} n_p \right). \tag{14}$$

Noticing that the derivative of the average number n_p is peaked at the Fermi impulsion p_F, derive the following expansion around p_F

$$U = \frac{V}{4\pi^2\hbar^3 m} \int_0^\infty \mathrm{d}p \left[p_F^6 + 6p_F^5(p - p_F) + \frac{30}{2!} p_F^4 (p - p_F)^2 \right] \frac{e^{\beta(\epsilon_p - \mu)}}{(e^{\beta(\epsilon_p - \mu)} + 1)^2}. \tag{15}$$

(2 points)

(c) At $T \ll T_F$, one may perform the approximation $p^2 - p_F^2 \simeq 2p_F(p - p_F)$. Show then that the internal energy is given by the formula

$$U = \frac{3}{5} N \epsilon_F \left[1 + \frac{5}{12}\pi^2 \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]. \tag{16}$$

Hint: For n an even integer

$$I_n = \int_{-\infty}^{+\infty} \mathrm{d}y \frac{y^n e^y}{(e^y + 1)^2} = 2n(n-1)!(1 - 2^{1-n})\zeta(n), \tag{17}$$

where ζ is Riemann's zeta function. We note that $I_n = 0$ if n is an odd integer. We also give that $\zeta(2) = \pi^2/6$.

(2 points)

(d) Show that the heat capacity of the Fermi gas follows the law

$$\frac{C_v}{Nk_B} = \frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}, \tag{18}$$

in the limit $T \ll T_F$.

(1 point)