21. White dwarves and the Chandrasekhar Mass (Bonus)

In exercise 23, we showed that the equation of state of a relativistic Fermi gas takes the form

$$P = C n_e^{\frac{1+n}{n}}, \tag{1}$$

where n_e is the density of electrons in the Fermi gas. Assuming that the mass inside a white dwarf corresponds to the same distribution, we expect the local pressure $P(r) = K(\rho(r))^{\frac{1+n}{n}}$ to stand at hydrostatic equilibrium, and to solve the equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P(r)}{\mathrm{d}r}\right) = -4\pi G\rho(r). \tag{2}$$

(a) In Eq.(2), replace the mass density $\rho(r)$ by the dimensionless function $\theta(r)$, which is defined by $\rho(r) = \rho_c\theta^n(r)$. Here we denote by $\rho_c = \rho(0)$ the mass density at the centre of the white dwarf. Furthermore, introduce the new coordinate $\xi = r/a$ via the scaling constant $a^2 = \frac{(n+1)K}{4\pi G}\rho_c^{(1-n)/n}$ and show that the differential equation (2) can be put into the following form

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{3}$$

What are the initial conditions for $\xi = 0$ that uniquely define the solution $\theta(\xi)$ of this differential equation?

(2 points)

(b) Justify why the first zero ξ_1 of the dimensionless mass density $\theta(\xi)$ determines the radius R of the white dwarf. Prove that its mass $M = M(R)$ is equivalently given by the equation

$$M(R) = 4\pi\rho_c R^3\left(-\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\Bigg|_{\xi=\xi_1}. \tag{4}$$

Hint: The numerical integration of the differential equation (3) yields the following values:

- Non-relivistic limit case: first zero $\xi_1^{nr} = 3.65$ and $\left(-\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\Big|_{\xi=\xi_1}^{nr} = 0.055642.$

- Ultrarelivistic limit case: first zero $\xi_1^{ur} = 6.90$ and $\left(-\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\Big|_{\xi=\xi_1}^{ur} = 0.006152.$

(c) Determine ρ_c as a function of R and use it to prove the relation

$$M(R) = 4\pi \left(\frac{4\pi G}{\xi_1^2 K(n+1)}\right)^{\frac{n}{1-n}} \left(-\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\Bigg|_{xi=xi_1} R^{\frac{3-n}{1-n}} \tag{5}$$

between the mass and radius of the white dwarf.

(1 point)

(d) Show that in the non-relativistic limit case the mass-radius relation (5) reads

$$\left(\frac{M}{M_\odot}\right) R^3 \approx (8887\text{km})^3, \tag{6}$$

where $M_\odot = 1.99 \times 10^30\text{kg}$ is the mass of the sun. So what happens to the radius R if we increase the mass M of the white dwarf?

(1 point)

(e) Show that the mass M of the white dwarf in the ultrarelativistic case does not depend on its radius R and determine for this case from (5) the Chandrasekhar limit mass

$$M_{\text{ch}} \approx 1.46 M_\odot. \tag{7}$$

What fate awaits a star after it has exhausted its hydrogen supply if its mass is greater than the Chandrasekhar mass?

(1 point)

22. Thermodynamic functions of the Bose gas

In the following, thermodynamic functions of the ideal Bose gas are to be derived. The equation of state in the limiting case of large volumes is given by

$$\frac{P}{k_B T} = \begin{cases} \frac{1}{\lambda^3} g_{5/2}(z), & (T > T_c) \\ \frac{1}{\lambda^3} g_{5/2}(1), & (T < T_c) \end{cases} \tag{8}$$

with T_c the critical temperature below which condensation occurs. The fugacity is defined for $\lambda^3/v < g_{3/2}(1)$ by the zero of the equation

$$\frac{\lambda^3}{v} = g_{3/2}(z). \tag{9}$$

For $\lambda^3/v > g_{3/2}(1)$, $z = 1$. Furthermore, Eq.(9) is equivalent to

$$\frac{g_{3/2}(z)}{g_{3/2}(1)} = \left(\frac{T_c}{T}\right)^{3/2}. \tag{10}$$

(a) Derive that the internal energy U takes the following form

$$\frac{U}{N} = \begin{cases} \frac{3}{2}\frac{k_B T v}{\lambda^3} g_{5/2}(z), & (T > T_c) \\ \frac{3}{2}\frac{k_B T v}{\lambda^3} g_{5/2}(1), & (T < T_c). \end{cases} \tag{11}$$

(1 point)

(b) Show that $g_{n-1}(z) = z\frac{\partial}{\partial z}g_n(z)$. Then, using Eqs. (9) and (10), show that the derivative of fugacity with respect to temperature satisfies the following relation

$$\frac{1}{z}\frac{\partial z}{\partial T} = -\frac{3}{2}\frac{\lambda^3}{Tv}\frac{1}{g_{1/2}(z)} = -\frac{3}{2}\frac{1}{T}\frac{g_{3/2}(z)}{g_{1/2}(z)}. \tag{12}$$

(1 point)

(c) Using Eqs. (11) and (12), determine the specific heat at constant volume C_V. The result is given by

$$\frac{C_V}{Nk_B} = \begin{cases} \frac{15}{4}\frac{v}{\lambda^3}g_{5/2}(z) - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}, & (T > T_c) \\ \frac{15}{4}\frac{v}{\lambda^3}g_{5/2}(1), & (T < T_c). \end{cases} \tag{13}$$

(2 points)

(d) Consider the limit $T \to \infty$ and show that $\frac{C_V}{Nk_B} \to \frac{3}{2}$. (1 Punkt)

(e) Now consider low temperatures $T \to 0$ and show that $\frac{C_V}{Nk_B} \propto T^{3/2}$. (1 point)

(f) We now want to investigate the discontinuity in the derivative of the specific heat at $T = T_c$. To do this, determine the derivative of the specific heat for $T < T_c$ and $T > T_c$ and show that

$$\left(\lim_{T\to T_c^+}\frac{\partial}{\partial T}\frac{C_V}{Nk_B}\right) - \left(\lim_{T\to T_c^-}\frac{\partial}{\partial T}\frac{C_V}{Nk_B}\right) = -\frac{27}{16\pi}\frac{\zeta(3/2)^2}{T_c} \approx -\frac{3.66}{T_c} \tag{14}$$

with the Riemann zeta function $\zeta(n)$.

Hint: The polylogarithms g_n have a well-defined limit for $z \to 1^-$ for $n > 1$ whereas $g_{1/2}(z)$ and $g_{-1/2}(z)$ diverge in the limit towards 1^-. For $n > 1$, the following relation holds for the value of the polylogarithms at $z = 1$

$$g_n(1) = \zeta(n),\ n > 1, \tag{15}$$

with the Riemann zeta function ζ. Furthermore, the following limit value may be used in the calculation

$$\lim_{z\to 1^-} \frac{g_{1/2}^3(z)}{g_{-1/2}(z)} = 2\pi. \tag{16}$$

(3 points)

23. Bose Gas in two dimensions

(a) Calculate the logarithm of the grandcanonical partition function in the limiting case

$$\lim_{V\to\infty} \frac{1}{V} \ln \mathcal{Q}(z, V, T) = \frac{1}{\lambda^2} g_2(z) \tag{17}$$

by using the following integral representation of $\ln \mathcal{Q}$ for large volumes

$$\ln \mathcal{Q} = -\frac{2\pi L^2}{h^2} \int_0^{+\infty} \mathrm{d}p p \ln(1 - ze^{-\beta\epsilon_p}), \tag{18}$$

where $V = L^2$ represents the surface available to the system.

(2 points)

(b) Calculate the mean number of particles per unit area as a function of z and T.

(2 points)

(c) Show that in the case of a two-dimensional Bose gas no Bose-Einstein condensation can take place and explain the result of your calculation physically.

Hint: Show that the integral for N diverges as the fugacity z approaches 1.

(4 points)