30. Boltzmann equation in the hydrodynamic regime and 0-th order solution approach

The aim of the exercise is to derive the hydrodynamic equations from the 0-th order solution of the Boltzmann equation. We assume that the free path length between the individual collisions should be much smaller than all other characteristic length scales, which leads to the fact that the gas should be locally in equilibrium and there satisfy a Maxwell-Boltzmann distribution. In this regime, therefore, $f(\vec{x},\vec{p};t) = f^{(0)}(\vec{x},\vec{p};t) + g(\vec{x},\vec{p};t)$ seems reasonable as a solution approach for the Boltzmann equation, where

$$f^{(0)} = (\vec{x},\vec{p};t) = \frac{\varrho(\vec{x},t)}{m} \cdot \left(\frac{1}{2\pi mk\,T(\vec{x},t)}\right)^{3/2} \exp\left[-\frac{(\vec{p}-m\vec{u}(\vec{x},t))^2}{2mk\,T(\vec{x},t)}\right]$$

and $g(\vec{x},\vec{p}_1;t)$ represents a small correction. In the local Maxwell-Boltzmann distribution $f^{(0)}$, $T(\vec{x},t)$ denotes the local temperature, $\varrho(\vec{x},t)$ the local mass density and $\vec{u}(\vec{x},t)$ the local mean velocity.

(a) Show by explicitly calculating the integrals that

(i) $m\int_{\mathbb{R}^3} f^{(0)}(\vec{x},\vec{p};t)\,d^3p = \varrho(\vec{x},t)\,,$

(ii) $m\int_{\mathbb{R}^3} \left(\frac{\vec{p}}{m}\right) f^{(0)}(\vec{x},\vec{p};t)\,d^3p = \varrho(\vec{x},t)\,\vec{u}(\vec{x},t)\,,$

(iii) $\varrho(\vec{x},t)\,\varepsilon(\vec{x},t) \equiv m\int_{\mathbb{R}^3} \frac{m}{2}\left(\frac{\vec{p}}{m}-\vec{u}\right)^2 f^{(0)}(\vec{x},\vec{p};t)\,d^3p = \frac{3}{2}\,\varrho(\vec{x},t)\,k\,T(\vec{x},t)\,,$

where $\varepsilon(\vec{x},t) = \langle \frac{m}{2}(\frac{\vec{p}}{m}-\vec{u})^2\rangle$ denotes the average thermal energy density.

(4 points)

(b) Using the definition of the Boltzmann equation, verify that for $f^{(0)}(\vec{x},\vec{p};t)$ the collision term

$$\left(\frac{\partial f^{(0)}}{\partial t}\right)_{\text{coll}} = \int_{\mathbb{R}^3} d^3p_2 \int_{\mathbb{R}^3} d^3p_1' \int_{\mathbb{R}^3} d^3p_2'\, W(\vec{p}_1,\vec{p}_2|\vec{p}_1',\vec{p}_2')\left(f_1^{(0)\prime} f_2^{(0)\prime} - f_1^{(0)}\, f_2^{(0)}\right)$$

vanishes, where the transition probability is

$$W(\vec{p}_1,\vec{p}_2|\vec{p}_1',\vec{p}_2') = |\mathcal{T}_{fi}|^2\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2')\,\delta\left(\frac{1}{2m}(|\vec{p}_1|^2+|\vec{p}_2|^2-|\vec{p}_1'|^2-|\vec{p}_2'|^2)\right).$$

(2 points)

(c) In the 0-th order approximation it is assumed that $f^{(0)}(\boldsymbol{x}, \boldsymbol{p}_1; t)$ is an approximate solution of the Boltzmann equation, i.e.

$$\frac{\partial f_1^{(0)}}{\partial t} + \sum_{i=1}^{3} \left[\frac{p_{1,i}}{m} \frac{\partial f_1^{(0)}}{\partial x_i} + F_i \frac{\partial f_1^{(0)}}{\partial p_{1,i}} \right] \approx 0\,.$$

Justify the approximate validity of the following relations

$$\frac{\partial \varrho}{\partial t} + \vec{\nabla}_{\vec{x}}(\varrho\,\vec{u}) \approx 0 \tag{1}$$

$$\varrho \left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k} \right) u_i \approx -\sum_{k=1}^{3} \frac{\partial P_{ik}}{\partial x_k} + \frac{\varrho}{m} F_i \tag{2}$$

$$\frac{\partial E}{\partial t} + \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \big(E\,u_k + \sum_{i=1}^{3} P_{ki}\,u_i + J_k \big) \approx \frac{\varrho}{m} \sum_{k=1}^{3} u_k\,F_k \tag{3}$$

where are denoted the energy density $E(\vec{x}, t) = \frac{\varrho}{2}\,\vec{u}^{\,2}(\vec{x}, t) + \frac{\varrho}{m}\,\varepsilon(\vec{x}, t)$, the pressure tensor

$$P_{ik}(\vec{x}, t) = \varrho\,\langle \big(\tfrac{p_i}{m} - u_i\big)\big(\tfrac{p_k}{m} - u_k\big) \rangle = m \int_{\mathbb{R}^3} \big(\tfrac{p_i}{m} - u_i\big)\big(\tfrac{p_k}{m} - u_k\big)\, f^{(0)}(\vec{x}, \vec{p}; t)\, d^3p$$

and the local heat current density

$$\begin{aligned} \vec{J}(\vec{x}, t) &= \frac{\varrho}{m} \left\langle \left(\frac{\vec{p}}{m} - \vec{u} \right) \frac{m}{2} \left(\frac{\vec{p}}{m} - \vec{u} \right)^2 \right\rangle \\ &= \int_{\mathbb{R}^3} \left(\frac{\vec{p}}{m} - \vec{u} \right) \frac{m}{2} \left(\frac{\vec{p}}{m} - \vec{u} \right)^2 f^{(0)}(\vec{x}, \vec{p}; t)\, d^3p. \end{aligned}$$

(2 points)

(d) By substituting the local Maxwell-Boltzmann distribution $f^{(0)}$ into the above expressions, prove that

(iv) $P_{ik}(\vec{x}, t) = P(\vec{x}, t)\,\delta_{ik}$ mit $P = \frac{\varrho}{m} kT$,

(v) $\vec{J}(\vec{x}, t) = 0\,.$

(3 points)

(e) Based on the equations (1)-(3), derive the following evolution equation for the mean thermal energy density:

$$\frac{\varrho}{m}\left(\frac{\partial}{\partial t}+\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\right)\varepsilon+\sum_{i=1}^{3}\frac{\partial J_i}{\partial x_i}\approx-\sum_{i,k=1}^{3} P_{ik}\frac{\partial u_i}{\partial x_k}\,.$$

(2 points)

(f) Using the relations (i)-(v), deduce that the 0-th order solution of the Boltzmann equation implies the hydrodynamic equations of motion (without dissipation):

$$\frac{\partial \varrho}{\partial t}+\vec{\nabla}_{\vec{x}}(\varrho\,\vec{u})\approx 0$$

$$\varrho\left(\frac{\partial}{\partial t}+\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\right)u_i\approx-\frac{\partial P}{\partial x_i}+\frac{\varrho}{m}F_i$$

$$\frac{\varrho}{m}\left(\frac{\partial}{\partial t}+\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\right)\varepsilon\approx -P\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$$

Do these equations of motion represent a closed system of equations with which the densities $\varrho(\vec{x},t)$, $\vec{u}(\vec{x},t)$ and $\varepsilon(\vec{x},t)$ can be uniquely determined?

(2 points)

31. The Maxwell-Boltzmann distribution function as the most probable distribution

If we are only interested in the properties of a gas in equilibrium, there is an alternative derivation of the Maxwell-Boltzmann distribution function in μ-space besides the access via the Boltzmann equation, which will be presented in this exercise.

We assume an ensemble of isolated systems with constant energy $\tilde{E}\in[E,E+\Delta]$. Let each of these systems consist of N distinguishable particles whose positions and momentum are given by $\boldsymbol{q}=(q_1,\ldots,q_{3N})$ and $\boldsymbol{p}=(p_1,\ldots,p_{3N})$ and which are located in a finite volume $V\subset\mathbb{R}^3$. There should be no external forces acting on the particles. Let the total phase space volume available to the microstates $(\boldsymbol{p},\boldsymbol{q})$ of the ensemble be denoted by $\mathcal{G}$. Furthermore, we assume that our system in equilibrium is described by the density function of the microcanonical ensemble (postulate of equal „a priori“ probability) $(\Delta \ll E)$

$$\rho_{\text{mic}}(\boldsymbol{p},\boldsymbol{q})=\begin{cases}\frac{1}{\Gamma(E)} & \forall\ (\boldsymbol{p},\boldsymbol{q})\in\mathbb{R}^{6N}\ \text{ mit }\ E<H(\boldsymbol{p},\boldsymbol{q})<E+\Delta\\ 0 & \text{sonst}\end{cases}$$

That is, all microstates $(\,\boldsymbol{p},\boldsymbol{q}\,)$ of the system for which the macroscopic „constraint“ $E < H(\boldsymbol{p},\boldsymbol{q}) < E + \Delta$ holds are equally likely to occur in the ensemble mean.

The one-particle distributionfunction $f(\boldsymbol{x},\boldsymbol{p})$ corresponding to $\rho_{\text{mic}}(\vec{p},\vec{q})$ in μ-space ($\boldsymbol{x},\boldsymbol{p} \in \mathbb{R}^3$) is difficult to calculate. For this reason, we take an alternative approach to the Maxwell-Boltzmann distribution in this task. First, we divide the μ-space accessible to the particles into small regions μ_i ($i \in \{1,\ldots K\}$, $K \gg 1$) whose positions we denote by $(\boldsymbol{p}_i,\boldsymbol{x}_i)$ and whose volume is $(\delta p)^3\,(\delta\, q)^3$. The number of particles in the i-th cell is then given by

$$n_i = \int_{\mu_i} f(\boldsymbol{x},\boldsymbol{p})\, d^3p\, d^3x$$

and in the limit case of infinitely small regions μ_i, the distribution function $f(\boldsymbol{x},\boldsymbol{p})$ can be approximated arbitrarily close by the occupation numbers n_i. The energy of a particle in the i-th cell is given by $\epsilon_i = \frac{\boldsymbol{p}_i^2}{2m}$ (interaction energy neglected because interaction radius $r_0 \ll \Delta\, q$). According to the premise, our system is isolated, so the occupation numbers n_i satisfy the following constraints:

$$\sum_{i=1}^{K} n_i = N \quad \text{und} \quad \sum_{i=1}^{K} n_i\,\epsilon_i = \tilde{E}\,. \tag{4}$$

(a) Assume that the gas is in a certain microstate $(\,\boldsymbol{p},\boldsymbol{q}\,)$. Are the values of the individual occupation numbers $n_i \in \mathbb{N}_0$ thereby uniquely determined? Conversely, is the microstate of the gas uniquely determined by a given occupation $\{n_i\}$? Justify with the help of a simple example.

(1 point)

(b) How many microstates exist at a given occupation $\{n_i\}$? Justify that the volume in phase space $\Omega\{n_i\} \in \mathcal{G}$ corresponding to a given occupation $\{n_i\}$ is given by the expression

$$\Omega\{n_i\} = C \cdot \frac{N!}{n_1!\, n_2! \ldots n_K!}.$$

The exact value of the proportionality constant C, which is independent of the n_i, is not of interest.

(2 points)

(c) We assume that the distribution function $f(\boldsymbol{x}, \boldsymbol{p})$ of the gas at equilibrium is the most probable distribution function. we assume that the occupation $\{\bar{n}_i\}$ associated to $f(\boldsymbol{x}, \boldsymbol{p})$ occupies the largest phase space volume in $\mathcal{G}$ $(\Omega\bar{n}_i\} \geq \Omega\{n_i\}$ for all occupations $\{n_i\}$ satisfying the constraints (4)). Determine the maximum of $\ln(\Omega\{n_i\})$ taking into account the constraints (4) using the Lagrange multipliers and thus show that

$$\bar{n}_i = \alpha \cdot e^{-\beta\epsilon_i}\,, \quad \alpha, \beta = \text{const.}\ .$$

In doing so, consider the occupation numbers $\bar{n}_i \gg 1$ as real numbers and use Stirling's formula in the lowest approximation $\ln n! \approx n \ln n - n$. In particular, show that it is a maximum of $\ln(\Omega\{n_i\})$ and not a minimum.

(3 points)

(d) Derive the Maxwell-Boltzmann distribution from this result

$$f(\boldsymbol{x}, \boldsymbol{p}) = \frac{n}{(2\pi\, mkT)^{3/2}}\, e^{-\frac{\boldsymbol{p}^2}{2mkT}}\,,$$

where $n = N/V$ denotes the constant particle density. Determine the constants α and β in analogy to the lecture on the stationary solution of the Boltzmann equation.

(3 points)