# Exercises for
# Theoretical physics IV

**WiSe 2022/23** **Sheet 8** **12.12.2022**

## Exercise 15 *Classical approximation of the quantum gas*

A gas of quantum particles of mass $m$ at a density $n$ and temperature $T$ can be approximated by a classical gas if the mean distance $d$ between the particles, defined as

$$d = \frac{1}{n^{1/3}}, \tag{1}$$

is much larger than the thermal de Broglie wavelength, namely $d \gg \lambda_{\mathrm{th}}$. For a gas of $N = 10^{19}$ particles confined in a volume $V = 1\ \mathrm{cm}^3$, determine the corresponding interparticle distance $d$ and compare it with the de Broglie wavelength of a hydrogen molecules at $T \approx 290$ K: $\lambda_{\mathrm{th}} \approx 0.72 \times 10^{-8}$ cm. Check whether the condition to treat the gas as a classical system is fulfilled or not.

*(1 point)*

## Exercise 16 *Entropic elasticity*

For a simple model of entropic elasticity, consider a one-dimensional chain consisting of $N$ elements. Each element of the chain has the length $a$ and should be able to align itself either parallel or antiparallel to the connection of the two end points. $n$ elements of the chain are aligned antiparallel to the connection of the two end points, the remaining $N - n$ elements are aligned parallel to it. Let the approximation $N \gg 1$, $n \gg 1$ and $N - n \gg 1$ be valid.

a) Calculate the number of microstates belonging to the state described above. Give a relationship between $N$, $n$ and the length $x$ of the chain.

*(1 point)*

b) How does the entropy of the system depend on the length of the chain? Show that the entropy decreases with the length of the chain, i.e. $\partial_x S < 0$. What does this mean for the chain, assuming that the system tries to adopt a configuration where entropy is maximal.

*(2 points)*

## Exercise 17 *Entropy of the ideal gas*

Consider a classical ideal gas of particles of mass $m$. The Hamiltonian function is then given by

$$H = \frac{1}{2m} \sum_{i=1}^{N} p_i^2, \tag{2}$$

with $p_i$ being the momentum of the $i$-th particle.

a) Determine the phase space volume $\Sigma(E)$ with

$$\Sigma(E) = \frac{1}{h^{3N}} \int_{H(p,q)<E} \mathrm{d}^{3N}q\mathrm{d}^{3N}p, \tag{3}$$

and from this determine the following formula for the entropy $S$ of the ideal gas

$$S = Nk_B \ln\left[V\left(\frac{E}{N}\right)^{3/2}\right] + \frac{3}{2}Nk_B\left[1 + \ln\left(\frac{4\pi m}{3h^2}\right)\right] \tag{4}$$

**Hint:** The surface of the $3N$ dimensional unit sphere is $2\frac{\pi^{3N/2}}{\Gamma(3N/2)}$.

*(4 points)*

b) Show that from the formula for entropy in Eq.(6) one can obtain the caloric equation of state $E = \frac{3}{2}k_BT$ and the thermal equation of state $pV = k_BNT$.

*(2 points)*

## Exercise 18 *Gibbs paradox*

We previously demonstrated that the entropy of an ideal gas was given by the formula

$$S = Nk_B \ln(Vu^{3/2}) + Ns_0, \tag{5}$$

where $u = \frac{3}{2}k_BT$ and $s_0 = \frac{3k_B}{2}\left(1 + \ln\frac{4\pi m}{3h^2}\right)$. Considering two ideal gases, with $N_1$ and $N_2$ particles respectively, kept in two seperate volumes $V_1$ and $V_2$ at the same temperature and same density.

a) Determine the difference in entropy of the combined system after the gases are allowed to mix in a joint volume $V = V_1 + V_2$, and show that this change of entropy is represented by

$$\frac{\Delta S}{k_B} = N_1 \ln\frac{V}{V_1} + N_2 \ln\frac{V}{V_2}, \tag{6}$$

which is the mixing entropy.

*(1 point)*

b) The Gibbs paradox arises when one considers the case of two identical ideal gases. Since the derivation of Eq. (5) is indenpendent of the identity of the gases, joining the two systems would also lead to an increase of the entropy. This is a disastrous result since it imples that the entropy of a gas depends on its history, and therefore cannot be only a function of the thermodynamics state of the gas. Gibbs solved this paradox by postulating that the expression of $\Sigma(E)$, the number of states below the energy $E$ is incomplete and should be divided by $N!$. Show that the entropy of the ideal gas then becomes

$$S = Nk_B \ln\left[\frac{V}{N}u^{3/2}\right] + \frac{3}{2}Nk_B\left[\frac{5}{3} + \ln\left(\frac{4\pi m}{3h^2}\right)\right]. \tag{7}$$

*(1 point)*

c) Work out the entropy of mixing for the case of different gases and then for identical gases and show that Eq. (6) solves the Gibbs paradox.

*(1 point)*