Übung zur Vorlesung
Theoretische Physik IV

WiSe 2022/23 **Sheet 12** **23.01.2023**

Exercise 27 *Thermodynamic functions of the Bose gas*

In the following, thermodynamic functions of the ideal Bose gas are to be derived. The equation of state in the limiting case of large volumes is given by

$$\frac{P}{k_B T} = \begin{cases} \frac{1}{\lambda^3} g_{5/2}(z), & (T > T_c) \\ \frac{1}{\lambda^3} g_{5/2}(1), & (T < T_c) \end{cases} \tag{1}$$

with T_c the critical temperature below which condensation occurs. The fugacity is defined for $\lambda^3/v < g_{3/2}(1)$ by the zero of the equation

$$\frac{\lambda^3}{v} = g_{3/2}(z). \tag{2}$$

For $\lambda^3/v > g_{3/2}(1)$, $z = 1$. Furthermore, Eq.(9) is equivalent to

$$\frac{g_{3/2}(z)}{g_{3/2}(1)} = \left(\frac{T_c}{T}\right)^{3/2}. \tag{3}$$

a) Derive that the internal energy U takes the following form

$$\frac{U}{N} = \begin{cases} \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(z), & (T > T_c) \\ \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(1), & (T < T_c). \end{cases} \tag{4}$$

(1 point)

b) Show that $g_{n-1}(z) = z \frac{\partial}{\partial z} g_n(z)$. Then, using Eqs. (2) and (3), show that the derivative of fugacity with respect to temperature satisfies the following relation

$$\frac{1}{z} \frac{\partial z}{\partial T} = -\frac{3}{2} \frac{\lambda^3}{T v} \frac{1}{g_{1/2}(z)} = -\frac{3}{2} \frac{1}{T} \frac{g_{3/2}(z)}{g_{1/2}(z)}. \tag{5}$$

(1 point)

c) Using Eqs. (4) and (5), determine the specific heat at constant volume C_V. The result is given by

$$\frac{C_V}{N k_B} = \begin{cases} \frac{15}{4} \frac{v}{\lambda^3} g_{5/2}(z) - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)}, & (T > T_c) \\ \frac{15}{4} \frac{v}{\lambda^3} g_{5/2}(1), & (T < T_c). \end{cases} \tag{6}$$

(2 points)

d) Consider the limit $T \to \infty$ and show that $\frac{C_V}{Nk_B} \to \frac{3}{2}$.

(1 point)

Now consider low temperatures $T \to 0$ and show that $\frac{C_V}{Nk_B} \propto T^{3/2}$.

(1 point)

e) We now want to investigate the discontinuity in the derivative of the specific heat at $T = T_c$. To do this, determine the derivative of the specific heat for $T < T_c$ and $T > T_c$ and show that

$$\left(\lim_{T\to T_c^+} \frac{\partial}{\partial T}\frac{C_V}{Nk_B}\right) - \left(\lim_{T\to T_c^-} \frac{\partial}{\partial T}\frac{C_V}{Nk_B}\right) = -\frac{27}{16\pi}\frac{\zeta(3/2)^2}{T_c} \approx -\frac{3.66}{T_c} \tag{7}$$

with the Riemann zeta function $\zeta(n)$.

Hint: The polylogarithms g_n have a well-defined limit for $z \to 1^-$ for $n > 1$ whereas $g_{1/2}(z)$ and $g_{-1/2}(z)$ diverge in the limit towards 1^-. For $n > 1$, the following relation holds for the value of the polylogarithms at $z = 1$

$$g_n(1) = \zeta(n), \quad n > 1, \tag{8}$$

with the Riemann zeta function ζ. Furthermore, the following limit value may be used in the calculation

$$\lim_{z\to 1^-} \frac{g_{1/2}^3(z)}{g_{-1/2}(z)} = 2\pi. \tag{9}$$

(3 points)

Exercise 28 *Bose Gas in two dimensions*

a) Calculate the logarithm of the grandcanonical partition function in the limiting case

$$\lim_{V\to\infty} \frac{1}{V} \ln \mathcal{Q}(z, V, T) = \frac{1}{\lambda^2} g_2(z) \tag{10}$$

by using the following integral representation of $\ln \mathcal{Q}$ for large volumes

$$\ln \mathcal{Q} = -\frac{2\pi L^2}{h^2} \int_0^{+\infty} \mathrm{d}p\, p \ln(1 - z e^{-\beta\epsilon_p}), \tag{11}$$

where $V = L^2$ represents the surface available to the system.

(2 points)

b) Calculate the mean number of particles per unit area as a function of z and T.

(2 points)

c) Show that in the case of a two-dimensional Bose gas no Bose-Einstein condensation can take place and explain the result of your calculation physically.

Hint: Show that the integral for N diverges as the fugacity z approaches 1.

(4 points)