Exercises for
Theoretical physics IV

WiSe 2022/23 **Sheet 7** **06.12.2022**

Exercise 15 *Equivalent definitions of entropy*

We consider a microcanonical ensemble with N identical particles, a volume V and an energy that takes values between $E - \Delta$ and E ($\Delta > 0$). Show that the following three formulae lead to an equivalent definition of the entropy S, differing only by an additive constant of order $\mathcal{O}(\ln N)$.

$$S = k_B \ln \Gamma(E) \tag{1a}$$
$$S = k_B \ln \omega(E) \tag{1b}$$
$$S = k_B \ln \Sigma(E). \tag{1c}$$

Here $\Gamma(E)$ is the phase space volume occupied by the totality of all microstates with

$$\Gamma(E) = \int_{E-\Delta}^{E} \mathrm{d}E' \mathrm{Tr}\{\delta(E' - \hat{H})\}, \tag{2}$$

$\Sigma(E)$ is the phase space volume bounded by the area of energy E with

$$\Sigma(E) = \mathrm{Tr}\{\theta(E - \hat{H})\}, \tag{3}$$

and $\omega(E)$ is the density of states of the system at energy E which is defined as

$$\omega(E) = \frac{\partial \Sigma(E)}{\partial E}. \tag{4}$$

Hint: You may assume that $\Delta \ll E$ and that the hyperbody described by the integrand in Eq.(2) is such that $\Sigma(E - \Delta)/\Sigma(E) \to 0$, for fixed Δ and $N \to \infty$.

(2 points)

Exercise 16 *Microcanonical description of paramagnets*

Consider again a system of N distinct spin-1/2 particles in a magnetic field B. The individual spins can only assume two energy values $E_- = +E_0/2$ or $E_+ = -E_0/2$, where $E_0 = 2\mu_B B$. Let n_- be the occupation of E_- and n_+ the occupation of E_+. Let the total energy of the system be U.

a) Determine the entropy of the system in the microcanonical ensemble.

(1 point)

b) Determine the temperature and show that it can take negative values. Then derive the expression for the internal energy U as a function of temperature.

(2 points)

c) Determine the expression for the specific heat, which is defined as

$$C = \left. \frac{\partial U}{\partial T} \right|_B . \tag{5}$$

(1 point)

d) Derive the expression for the total magnetisation $M^z = \mu_B(n_+ - n_-)$, and then derive the form of the magnetic susceptibility χ given by

$$\chi = \left. \frac{\partial M^z}{\partial B} \right|_T \tag{6}$$

is given. Then show that in the limiting case $T \gg \mu_B B/k$ the susceptibility follows Curie's law, i.e. $\chi \sim 1/T$.

(2 points)