Contents:

- Schmidt decomposition (N, P)
  - Ex. (N.2.79)
- Teleportation (N)
  - original example
  - Alice result lost (information is physical & Schmidt decomp.)

Learning goals:

- Quantify entanglement via Schmidt rank
- Obtain intuition why information is physical

Schmidt decomposition (N. 2.5; P 2.4; W 3.8):

When considering pure states on composite systems $H_A \otimes H_B$, the Schmidt decomposition is a powerful tool to analyze entanglement between the two subsystems (also heavily used in state-of-the-art computational physics in tensor network methods).

Theorem N 2.7:

Suppose $|\psi\rangle$ is a pure state of the composite system $H_A \otimes H_B$. Then there exists an orthonormal basis $|i\rangle_A$ for system A, and an orthonormal basis $|i\rangle_B$ for system B such that

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i \, |i\rangle_A |i\rangle_B ,$$

where λ_i are strictly positive real numbers satisfying

$$\sum_{i=1}^{d} \lambda_i^2 = 1 .$$

The λ_i are called Schmidt coefficients and $d \leq \min\{\dim(H_A), \dim(H_B)\}$ is the Schmidt rank.

Proof (taken from W.3.8):

This is a restatement of the singular value decomposition (SVD) of a matrix: $a = u\, d\, v^\dagger$

$$\underbrace{a}_{m\times n} = \underbrace{u}_{m\times m}\ \underbrace{d}_{m\times n}\ \underbrace{v^\dagger}_{n\times n}$$

M is $m\times n$ complex matrix

U, V are unitary, Σ is real diagonal

usually chosen $\sigma_1 \geq \sigma_2 \geq \dots$

$$d = \begin{array}{c} n \\ \left[\begin{array}{ccc} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \\ \hline & 0 & \end{array}\right] \end{array} m$$

The diagonal entries Σ_{ii} are uniquely determined by a and are called the singular values.

A general pure state $|\psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed as

$$|\psi\rangle = \sum_{\mu=1}^{d_A} \sum_{\nu=1}^{d_B} a_{\mu\nu}\, |\mu\rangle_A |\nu\rangle_B$$

By the SVD $a = u\, d\, v^\dagger$ with unitary u, v and diagonal d with $d_{ii} \geq 0$.

Inserting $a_{\mu\nu} = \sum_{i=1}^{d} u_{\mu i}\, d_{ii} \underbrace{(v^+)_{i\nu}}_{= v^*_{\nu i}}$, we find

$$|\psi\rangle = \sum_{\mu=1}^{d_A} \sum_{\nu=1}^{d_B} a_{\mu\nu} \, |\mu\rangle_A |\nu\rangle_B =$$

$$= \sum_{i=1}^{d} \sum_{\mu=1}^{d_A} \sum_{\nu=1}^{d_B} u_{\mu i}\, d_{ii}\, v^*_{\nu i} \, |\mu\rangle_A |\nu\rangle_B .$$

Defining new bases: $|i\rangle_A = \sum_{\mu=1}^{d_A} u_{\mu i} |\mu\rangle_A$,

$|i\rangle_B = \sum_{\nu=1}^{d_B} \underbrace{(v^+)_{i\nu}}_{= v^*_{\nu i}} |\nu\rangle_B$ and $d_{ii} = \lambda_i$, this becomes

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i \, |i\rangle_A |i\rangle_B .$$

Since U and V are unitary, the states $|i\rangle_A, |i\rangle_B$ are orthonormal (U, V are basis transformations).

The number nonzero Schmidt coefficients λ_i equals the rank of a. Usually one sorts them in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$. The Schmidt form is also called standard form for $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Useful conclusions from Schmidt decomposition:

- Reduced density matrix:

Since $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i \, |i\rangle_A |i\rangle_B$$

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \sum_{i,j=1}^{d} \lambda_i \lambda_j \underbrace{|i\rangle_A |i\rangle_B \langle j|_A \langle j|_B}_{\text{also written as } |i\rangle_A\langle j|_A \otimes |i\rangle_B\langle j|_B}$$

The reduced density matrices are then

$$\rho_A = \mathrm{Tr}_B\, \rho = \sum_{i,j=1}^{d} \lambda_i \lambda_j \, |i\rangle_A\langle j|_A \otimes \underbrace{\langle j|i\rangle_B}_{=\delta_{ij}} =$$

$$= \sum_{i=1}^{d} \lambda_i^2 \, |i\rangle_A\langle i|_A \, .$$

Similarly, $\rho_B = \sum_{i=1}^{d} \lambda_i^2 \, |i\rangle_B \langle i|_B$.

$\Rightarrow$ Schmidt bases $|i\rangle_A$, $|i\rangle_B$ are eigenbases of the reduced density matrices ρ_A and ρ_B.

$\Rightarrow$ $\lambda_i^2 = p_i \Leftrightarrow \lambda_i = \sqrt{p_i}$

are eigenvalues of the reduced density matrices ρ_A and ρ_B.

Note that ρ_A and ρ_B have the same nonzero eigenvalues. If $\dim(\mathcal{H}_A) \neq \dim(\mathcal{H}_B)$, the number of zero eigenvalues differs.

$\Rightarrow$ Purities $\mathrm{Tr}_A(\rho_A^2) = \mathrm{Tr}_B(\rho_B^2)$ are the same.

- Schmidt coefficients λ_i describe entanglement between A and B:

$S_{VN} = -\mathrm{Tr}\, \rho \ln \rho$ (von-Neumann entanglement entropy, describes entanglement of mixed state after Tr_B operation)

For reduced density matrices

$S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A = -\sum_{i=1}^{d} p_i \ln p_i =$

$$= -\sum_{i=1}^{d} \lambda_i^2 \ln \lambda_i^2 .$$

Entanglement spectrum $\{\varepsilon_i\}$ defined as $\lambda_i^2 = e^{-\varepsilon_i}$.

Thus, a bipartite pure state is entangled if $d > 1$. Otherwise, iff $d = 1$, the pure state is separable $|\psi\rangle = |\psi_A\rangle |\psi_B\rangle$.

- Schmidt number d is preserved under unitary tf on system A or system B alone:

If $$|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B ,$$

then $$\mathcal{U}_A |\psi\rangle = \sum_i \lambda_i \left(\mathcal{U}_A |i\rangle_A\right) |i\rangle_B .$$

acts only on A

Note. If ρ_A (and thus ρ_B) have no degenerate nonzero eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_r > \lambda_{r+1} = 0 = \lambda_{r+2} = \ldots$, the Schmidt decomposition is uniquely determined by ρ_A and ρ_B: diagonalize ρ_A using $|i\rangle_A$ and ρ_B using $|i\rangle_B$ and pair up states with the same λ_i:

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |i\rangle_A |i\rangle_B .$$

We can only apply $U(1)$ phase on $|i\rangle_A \to e^{i\phi}|i\rangle_A$ and opposite phase on $|i\rangle_B \longrightarrow e^{-i\phi}|i\rangle_B$.

However, if λ_i's are degenerate, we can apply simultaneous unitary tf V within the degenerate subspace $(\lambda_{i_0} = \dots = \lambda_{i_0+l})$

$$|i\rangle_A = \sum_{i'=i_0}^{i_0+l} V_{i'i}\,|i'\rangle_A$$

$$|i\rangle_B = \sum_{i''=i_0}^{i_0+l} (V^+)_{ii''}\,|i''\rangle_B\,.$$

This part of the summation then reads

$$\lambda_{i_0}\sum_{i=i_0}^{i_0+l} |i\rangle_A\,|i\rangle_B = \lambda_{i_0}\underbrace{\sum_{i=i_0}^{i_0+l}\sum_{i',i''} V_{i'i}\,(V^+)_{ii''}}_{\sum_i V_{i'i}(V^+)_{ii''} = \delta_{i'i''}}\,|i'\rangle_A\,|i''\rangle_B$$

$$= \lambda_{i_0}\sum_{i'=i_0}^{i_0+l} |i'\rangle_A\,|i'\rangle_B\,.$$

In presence of degenerate Schmidt coefficients, the is thus an ambiguity in the choice of Schmidt basis (can simultaneously rotate in degenerate subspace)

Let's close this section with two examples (do example in teleportation section)

① $|\psi\rangle = \frac{1}{\sqrt{2}}\left[\,|00\rangle + |11\rangle\,\right] \Rightarrow \lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$.

already in Schmidt form

$= \sum_{\mu,\nu=0}^{1} a_{\mu\nu} \, |\mu\rangle_A \, |\nu\rangle_B$

$|i\rangle_A = \{ |0\rangle, |1\rangle \}$

$|i\rangle_B = \{ |0\rangle, |1\rangle \}$

$$\Rightarrow \quad a = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

SVD: $\quad a = u\, d\, v^{+}$

$$u = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = v^{+} \quad , \quad d = \begin{pmatrix} 1/\sqrt{2} & \\ & 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \rho_B .$$

② (HW)

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left[|00\rangle + |01\rangle + |10\rangle \right]$$

$$\Rightarrow a = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \end{pmatrix}$$

SVD : $a = u\, d\, v^{+} \Rightarrow \lambda_0 = \sqrt{\frac{1}{6}(3+\sqrt{5})}$

$$\lambda_1 = \sqrt{\frac{1}{6}(3-\sqrt{5})}$$

$$u = \begin{pmatrix} 0.85 & 0.53 \\ 0.53 & -0.85 \end{pmatrix} \Rightarrow |i=0\rangle_A = 0.85\, |\overset{u_{11}}{0}\rangle_A + 0.53\, |1\rangle_A$$

$$|i=1\rangle_A = 0.53\, |0\rangle_A - 0.85\, |1\rangle_A$$

$$v = \begin{pmatrix} 0.85 & -0.53 \\ 0.53 & 0.85 \end{pmatrix} \quad |i=0\rangle_B = 0.85\, |\overset{v_{11}}{0}\rangle_B + 0.53\, |1\rangle_B$$

$$|i=1\rangle_B = -0.53\, |0\rangle_B + 0.85\, |1\rangle_B$$

$$\Rightarrow |\psi\rangle = \sqrt{\frac{1}{6}(3+\sqrt{5})}\; |i=0\rangle_A\, |i=0\rangle_B$$

$$+ \sqrt{\frac{1}{6}(3-\sqrt{5})}\; |i=1\rangle_A\, |i=1\rangle_B$$

<u>Check</u>: $0.85 = a$, $0.53 = b$

$$\lambda_0 \left[a^2 |00\rangle + b^2 |11\rangle + ab |01\rangle + ab |10\rangle \right]$$

$$+ \lambda_1 \left[-b^2 |00\rangle - a^2 |11\rangle + ab |01\rangle + ab |10\rangle \right] =$$

$$= \underbrace{\left(\lambda_0 a^2 - \lambda_1 b^2 \right)}_{= 1/\sqrt{3}} |00\rangle + \underbrace{\left(\lambda_0 b^2 - \lambda_1 a^2 \right)}_{= 0} |11\rangle$$

$$+ \underbrace{\left(\lambda_0 + \lambda_1 \right)}_{= 1/\sqrt{3}} ab \left(|01\rangle + |10\rangle \right)$$

✓

<u>Purification</u>:

Suppose we are given a state ρ_A of system H_A.

Spectral decomposition: $\rho_A = \sum_{i=1}^{r} p_i |i\rangle_A \langle i|_A$.

We can purify this state by introducing another „reference" system H_R with dimension at least r (but can be larger) and orthonormal basis $|i\rangle_R$.

The <u>pure state</u> $|AR\rangle = \sum_{i=1}^{r} \sqrt{p_i} |i\rangle_A |i\rangle_R$

has reduced density matrix $\rho_A = \mathrm{Tr}_R\left(|AR\rangle\langle AR|\right)$.

It is thus a purified version of the state ρ_A on an extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_R$.

Explicitly:

$$\mathrm{Tr}_R\left(|AR\rangle\langle AR|\right) = \sum_{i=1}^{r}\sum_{j=1}^{r} \sqrt{p_i}\sqrt{p_j}\,|i\rangle_A\langle j|_A \cdot \underbrace{\langle i|j\rangle_R}_{=\delta_{ij}} =$$

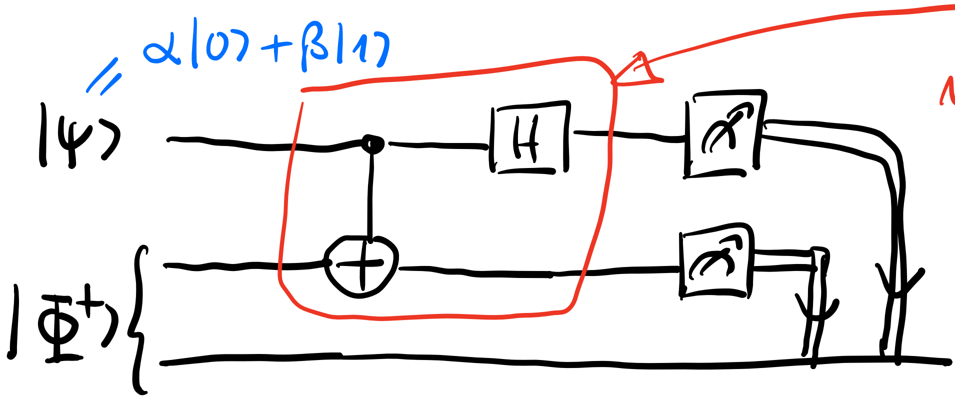
$$= \sum_{i=1}^{r} p_i\,|i\rangle_A\langle i|_A = \rho_A .$$

$|AR\rangle$ is a purification of ρ_A. We obtained $|AR\rangle$ as a pure state whose Schmidt basis for system A is the eigenbasis of ρ_A.

Teleportation & information is physical

Alice & Bob are separated in space & can communicate only classically. However, Alice wants to teleport a qubit in (unknown) state $|\psi\rangle$ to Bob.

Fortunately they shared an $\underbrace{\text{entangled Bell pair}}_{\text{also called ebit}}$ a while ago when they were together.



this circuit performs a measurement in the Bell basis, i.e. it maps

$$|\Phi^+\rangle \to |00\rangle$$
$$|\Phi^-\rangle \to |10\rangle$$
$$|\Psi^+\rangle \to |01\rangle$$
$$|\Psi^-\rangle \to |11\rangle$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$$

Explicitly:

$$|\Phi^+\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|00\rangle + |10\rangle\right] \xrightarrow{H} |00\rangle$$
$$|\Phi^-\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|00\rangle - |10\rangle\right] \xrightarrow{H} |10\rangle$$
$$|\Psi^+\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|01\rangle + |11\rangle\right] \xrightarrow{H} |01\rangle$$
$$|\Psi^-\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|01\rangle - |11\rangle\right] \xrightarrow{H} |11\rangle$$

State evolution during teleportation protocol:

$$|\psi\rangle|\Phi^+\rangle = \left[\alpha|0\rangle + \beta|1\rangle\right]\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$$\Big\downarrow \text{CNOT (qubits} = 01)$$

$$\frac{1}{\sqrt{2}}\left[\alpha\left(|000\rangle + |011\rangle\right) + \beta\left(|110\rangle + |101\rangle\right)\right.$$

$$\Big\downarrow H \text{ (qubit } 0)$$

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$$\frac{1}{2}\Big[\alpha\left(|000\rangle + |100\rangle + |011\rangle + |111\rangle\right)$$
$$+\beta\left(|010\rangle - |110\rangle + |001\rangle - |101\rangle\right)\Big] =$$

$$= \frac{1}{2}\Big[|00\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + |01\rangle\left(\alpha|1\rangle + \beta|0\rangle\right)$$

$$+ |10\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + |11\rangle\left(\alpha|1\rangle - \beta|0\rangle\right)\Big].$$

Now Alice measures her two qubits in the z basis and finds one of the four possible bitstrings $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, each with probability $p = \frac{1}{4}$.

The state of Bob's qubit depends on Alice's measurement outcome. Once she communicates her result (two classical bits), he can generate the state $|\psi\rangle$ on his qubit.

Specifically,

Alice's measurement result:	Bob's qubit	Operation to recover $\|\psi\rangle = \alpha\|0\rangle + \beta\|1\rangle$
$\|00\rangle$:	$\alpha\|0\rangle + \beta\|1\rangle$	I
$\|01\rangle$:	$\alpha\|1\rangle + \beta\|0\rangle$	X
$\|10\rangle$:	$\alpha\|0\rangle - \beta\|1\rangle$	Z
$\|11\rangle$:	$\alpha\|1\rangle - \beta\|0\rangle$	ZX

Since Alice must communicate her result via a classical communication channel, causality is not violated.

In fact, let's assume Alice cannot send her result to Bob. Then, his system is described by

$$\rho_{Bob} = \mathrm{Tr}_{Alice}\, \rho \,.$$

To get Bob's reduced density matrix, let us perform a Schmidt decomposition on the pure state before Alice's measurement:

$$|\psi\rangle = \frac{1}{2}\Big[|00\rangle\big(\alpha|0\rangle+\beta|1\rangle\big) + |01\rangle\big(\alpha|1\rangle+\beta|0\rangle\big)$$

$$+ |10\rangle\big(\alpha|0\rangle-\beta|1\rangle\big) + |11\rangle\big(\alpha|1\rangle-\beta|0\rangle\big)\Big]$$

$$= \sum_i \lambda_i \, |i\rangle_{Alice} \underbrace{|i\rangle_{Bob}}_{\{|0\rangle,\,|1\rangle\}}$$

SVD of matrix a defined as

$$|\psi\rangle = \sum_{\mu=1}^{d_A \overset{4}{=}} \sum_{\nu=1}^{d_B \overset{2}{=}} a_{\mu\nu}\, |\mu\rangle_A |\nu\rangle_B$$

$$a = \frac{1}{2}\begin{array}{c} \\ |00\rangle_A \\ |01\rangle_A \\ |10\rangle_A \\ |11\rangle_A \end{array}\begin{array}{c} \begin{array}{cc} |0\rangle_B & |1\rangle_B \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \\ \alpha & -\beta \\ -\beta & \alpha \end{pmatrix} \end{array} = \underset{4\times 4}{u}\ \underset{4\times 2}{d}\ \underset{2\times 2}{v^{+}}$$

$$d = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \multicolumn{2}{c}{0} \end{pmatrix} \quad \text{with } \lambda_1 = \lambda_2 = 1/\sqrt{2} \quad \left(= \sqrt{p_1} = \sqrt{p_2}\right)$$

$$v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{array}{l} |i=0\rangle_B = |\mu=1\rangle_B \\ |i=1\rangle_B = |\mu=0\rangle_B \end{array}$$

(can also choose $|0\rangle = |\mu=0\rangle$, $|1\rangle = |\mu=1\rangle$ as $\lambda_1 = \lambda_2$.)

$$u = \begin{pmatrix} \beta/\sqrt{2} & \alpha/\sqrt{2} & & \\ \alpha/\sqrt{2} & \beta/\sqrt{2} & * & * \\ -\beta/\sqrt{2} & \alpha/\sqrt{2} & & \\ \alpha/\sqrt{2} & -\beta/\sqrt{2} & & \end{pmatrix}$$

$$\Rightarrow |i=0\rangle_A = \frac{1}{\sqrt{2}}\left[\beta|00\rangle_A + \alpha|01\rangle_A - \beta|10\rangle_A + \alpha|11\rangle_A\right]$$

$$|i=1\rangle_A = \frac{1}{\sqrt{2}}\left[\alpha|00\rangle_A + \beta|01\rangle_A + \alpha|10\rangle_A - \beta|11\rangle_A\right]$$

We can check that these are orthonormal states (part of an orthonormal basis:

$$\langle i=1|_A\, i=0\rangle_A = \frac{1}{2}\left[\alpha^*\beta + \beta^*\alpha - \alpha^*\beta - \beta^*\alpha\right] =$$

$$= 0$$

$$\langle i=0|_A\, i=0\rangle_A = \frac{1}{2}\left[2|\alpha|^2 + 2|\beta|^2\right] = |\alpha|^2 + |\beta|^2 = 1$$

The Schmidt decomposition thus reads

$$|\psi\rangle = \frac{1}{\sqrt{2}}|i=1\rangle_A|0\rangle_B + \frac{1}{\sqrt{2}}|i=0\rangle_A|1\rangle .$$

The reduced density matrix of Bob's system is

$$\rho_B = \mathrm{Tr}_A\left[|\psi\rangle\langle\psi|\right] = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{I}{2}$$

Fully mixed state. Before/without receiving Alice's measurement result, Bob has a fully mixed state that does not contain any information on the state $|\psi\rangle$ (Schmidt eigenvalues are independent of α, β).

<u>Conclusion</u>: Information is physical. Bob's state with and without knowledge of Alice's result is different.

By receiving Alice's measurement outcomes Bob can recover a pure state $|\psi\rangle\langle\psi|$ from the mixture $\frac{I}{2}$ by applying the (one) appropriate gate operation $\in \{I, X, Z, ZX\}$.