Contents:

Learning goals:

- Quantum channels (continued)
  - Reversibility of channels
  - Adjoint of channels
  - Channel-state duality (Choi-Jamiołkowski isomorphism)

Operator-sum representation of a quantum channel:

$\rho \in \mathcal{H}_A$: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $|\psi_i\rangle \in \mathcal{H}_A$.

Evolution of ρ is CPTP map (= quantum channel):

$$\mathcal{E}(\rho) = \sum_{a=1}^{d \leq d_A^2} M_a \rho M_a^\dagger \quad \text{with} \quad \sum_a M_a^\dagger M_a = I.$$

- $\mathcal{E}(\rho) \geq 0$ if $\rho \geq 0$, also $(\mathcal{E} \otimes I_E) \geq 0$ for any E.
- $\mathrm{Tr}[\mathcal{E}(\rho)] = 1$
- Kraus operators are not unique: $N_\mu = \sum_a V_{\mu a} M_a$ unitary
- Number of Kraus operators $d \leq d_A^2$, where $d_A = \dim(A)$. We also write $d \equiv a_{max}$ when summing over a.
- Iff $d = 1$ the evolution is unitary. For $d \geq 2$ the system A gets entangled with E under $\mathcal{U}_{AE} : |\psi\rangle \otimes |0\rangle_E = \sum_a M_a |\psi\rangle |a\rangle_E$ and Tr_E results in a mixed state $\mathcal{E}(\rho)$.

- Composition of two channels $\mathcal{E}_1$ (described by $\{M_a\}$) and $\mathcal{E}_2$ (described by $\{N_b\}$):

$$\mathcal{E}_2[\mathcal{E}_1(\rho)] = \sum_{b=1}^{b_{max}} \sum_{a=1}^{a_{max}} N_b M_a \rho M_a^\dagger N_b^\dagger .$$

The composed channel $\mathcal{E}_2 \circ \mathcal{E}_1$ thus has $a_{max} \cdot b_{max}$ Kraus operators $\{ N_b M_a \}$.

Reversibility of a quantum channel mapping $A \to A$:

If a quantum channel $\mathcal{E}_1 : A \to A$ can be reversed, it holds

$$\mathcal{E}_2 \circ \mathcal{E}_1 (|\psi\rangle\langle\psi|) = \sum_{a,b} N_b M_a |\psi\rangle\langle\psi| M_a^\dagger N_b^\dagger = |\psi\rangle\langle\psi|$$

for any pure state $|\psi\rangle$.

LHS is sum of positive terms, the equality can only hold if each term is proportional to $|\psi\rangle\langle\psi|$ (other elements cannot be cancelled):

$$\Rightarrow \quad N_b M_a = \lambda_{ba} I \qquad \text{for each } a, b.$$

Using $\sum_b N_b^\dagger N_b = I$, we find

(Note that $a_{max} \neq b_{max}$ such that λ_{ba} is a rectangular matrix.)

$$M_b^\dagger M_a = M_b^\dagger \Big(\sum_c N_c^\dagger N_c\Big) M_a =$$

$$= \sum_c \underbrace{M_b^\dagger N_c^\dagger}_{=\lambda_{cb}^* I} \underbrace{N_c M_a}_{=\lambda_{ca} I} =$$

$$= \sum_c \lambda_{cb}^* \lambda_{ca} I = \beta_{ba} I$$

with $\beta_{ba} = \sum_c \lambda_{cb}^* \lambda_{ca}$.

Note that $\beta_{aa} = \sum_c |\lambda_{ca}|^2 > 0$ unless $M_a = 0$.

Here we consider quantum channels $\mathcal{E}_i : A \to A$

$\Rightarrow$ M_a, N_b are square $d_A \times d_A$ matrices.

They allow for a polar decomposition (N.2.1.10)

$$M_a = \underset{\uparrow}{U_a} \sqrt{M_a^\dagger M_a} \qquad \Big(\text{and } N_b = V_b \sqrt{N_b^\dagger N_b}\Big).$$

unitary U_a unique positive operator

(unique if M_a is invertible)

$$\Rightarrow \quad M_a = U_a \sqrt{M_a^\dagger M_a} = \sqrt{\beta_{aa}}\, U_a$$

And thus

$$M_b^\dagger M_a = U_b^\dagger U_a \sqrt{M_b^\dagger M_b} \sqrt{M_a^\dagger M_a} =$$

$$= U_b^\dagger U_a \sqrt{\beta_{bb}\,\beta_{aa}} = \beta_{ba}\, I$$

from above

Therefore, for each a,b for which $M_a \neq 0$ and $M_b \neq 0$:

$$U_a = \frac{\beta_{ba}}{\sqrt{\beta_{bb}\,\beta_{aa}}}\, U_b$$

Therefore, demanding reversibility of the channel requires that each Kraus operator M_a is proportional to a single unitary matrix and $\mathcal{E}_1$ is thus a unitary map.

<u>Conclusion</u>: A quantum channel $\mathcal{E}: A \to A$ can be inverted by another quantum channel only if it is unitary.

Makes sense: for $a_{max} > 1$ the U_{AE} entangles A with E and Tr_E results in a loss of information into the environment. If we do not observe or control E this information cannot be recovered.

<u>Note</u>: this argument applies only to a quantum channel that maps $A \to A$ of the same dimension. If $\mathcal{E} : A \to A'$ with $\dim(A') > \dim(A)$, the argument can be evaded as the Kraus operators are then rectangular and do not allow for a polar decomposition. This is essential when developing quantum error correction, where the addition of ancillary qubits allow to invert action of a quantum channel on A.

Adjoint of quantum channel ≙ Heisenberg picture of channels :

In Schrödinger picture the state ρ evolves :

$$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger$$

Hilbert-Schmidt inner product of operators

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B) .$$

for linear operators $A, B \in \mathcal{L}(\mathcal{H})$ (i.e. $A, B : \mathcal{H} \to \mathcal{H}$)

This allows us to define the adjoint map $\mathcal{E}^\dagger$ of channel $\mathcal{E}$.

Example. $\mathrm{Tr}(A \, \mathcal{E}(\rho))$ to compute expectation value of operator A.

Adjoint map $\mathcal{E}^\dagger : \mathcal{L}(H_B) \longrightarrow \mathcal{L}(H_A)$

to channel $\mathcal{E} : \mathcal{L}(H_A) \to \mathcal{L}(H_B)$ is defined as

$$\boxed{\mathrm{Tr}[\mathcal{E}^\dagger(A) \rho] = \mathrm{Tr}[A \, \mathcal{E}(\rho)].}$$

(or $\langle A, \mathcal{E}(B) \rangle = \langle \mathcal{E}^\dagger(A), B \rangle$.

Therefore,

$$\mathrm{Tr}[A\,\mathcal{E}(\rho)] = \mathrm{Tr}[\sum_a A M_a \rho M_a^\dagger] =$$

$$= \mathrm{Tr}[\sum_a M_a^\dagger A M_a \rho] \Rightarrow \boxed{\mathcal{E}^\dagger(A) = \sum_a M_a^\dagger A M_a}$$

cyclicity of Tr

with $\sum_a M_a^\dagger M_a = I$.

- Adjoint map of a channel is completely positive, but may <u>not</u> be trace preserving.

It is only trace preserving if $\sum_a M_a M_a^\dagger = I$.

- Adjoint map $\mathcal{E}^\dagger$ is <u>unital</u>, i.e., it preserves the identity:

$$\mathcal{E}^\dagger(I) = I$$

This follows from completeness $\sum_a M_a^\dagger M_a = I$ as

$$\mathcal{E}^\dagger(I) = \sum_a M_a^\dagger I M_a = I.$$

- If $\mathcal{E}^\dagger(\rho)$ is trace preserving, i.e.,

$\sum_a M_a M_a^\dagger = I$, then $\mathcal{E}^\dagger(\rho)$ describes a unital channel.

Unital channels map the maximally mixed density operator $\rho = \frac{I}{n}$ to itself.

[This is the quantum version of a doubly stochastic classical map, which maps probability distributions to probability distributions and preserves the uniform distribution.]

Quantum instruments:

We mentioned previously when introducing channels that we could keep record of some of the measurement results of the environment. Rather than performing Tr_E or keeping all results (= corresponding to a generalized measurement), we only keep some.

Labeling Kraus operators as $M_{a\mu}$ such that

$$\sum_{a,\mu} M^{+}_{a\mu} M_{a\mu} = I$$, we remember a but forget μ.

The post measurement state of

$$\mathcal{U}_{AE} : |\psi\rangle|0\rangle_E \mapsto \sum_{a,\mu} M_{a\mu} |\psi\rangle |a,\mu\rangle_E$$

is then

$$\mathcal{E}_a(\rho) = \sum_\mu M_{a\mu} \rho M_{a\mu}^\dagger$$

with probability $p(a) = \mathrm{Tr}\, \mathcal{E}_a(\rho)$.

Now, $\sum_\mu M_{a\mu}^\dagger M_{a\mu} \leq I$

is not neccessarily trace preserving.

We thus need to renormalize the state

$$\rho \longmapsto \frac{\mathcal{E}_a(\rho)}{\mathrm{Tr}[\mathcal{E}_a(\rho)]} \quad \text{(nonlinear evolution).}$$

After sequence of measurements with outcomes $\{a_1, \ldots, a_n\}$,

the state is

$$\rho \longmapsto \frac{\mathcal{E}_{a_n} \circ \mathcal{E}_{a_{n-1}} \circ \ldots \circ \mathcal{E}_{a_1}(\rho)}{\mathrm{Tr}[\mathcal{E}_{a_n} \circ \ldots \circ \mathcal{E}_{a_1}(\rho)]} .$$

Placing the measurement outcome in a separate register, this can also be written as

$$\rho \longmapsto \sum_a \mathcal{E}_a(\rho) \otimes |a\rangle\langle a|$$

quantum output

classical output

This is called a quantum instrument.

Channel-state duality:

Have seen that tracing out environment of unitary map U_{AE} on extend Hilbert space $U_{AE}: |\psi\rangle|0\rangle_E \longmapsto \sum_a M_a|\psi\rangle|a\rangle_E$ yields CPTP map (= quantum channel) on $\mathcal{E}: A \to A'$. It thus maps density operators to density operators.

The reverse is also true: every CPTP map has a unitary realization on an extended system.

Thus:

Every CPTP map has operator-sum representation and a unitary realization U_{AE} on an extended system.

We will construct the corresponding unitary map explicitly below (using Stinespring dilation). First, let us show a duality between CPTP maps $\mathcal{E}: A \to A'$ and states $\rho_{EA'} \geq 0$.

This will allow us to answer the question:

How many quantum channels $\mathcal{E}: A \to A'$ exist?

Consider reference system H_E with $\dim(H_E) = \dim(H_A) = d$.

Let's introduce the maximally entangled state

$$|\tilde{\Phi}\rangle_{EA} = \sum_{i=0}^{d-1} \underbrace{|i\rangle_E}_{\text{orthonormal basis of } H_E} \otimes \underbrace{|i\rangle_A}_{\text{orthonormal basis of } H_A}$$

Note that $|\tilde{\Phi}\rangle_{EA}$ has norm $\sqrt{d}$. This unconventional normalization avoids annoying factors of d in formulas below.

Since $\mathcal{E}: A \to A'$ is completely positive, it maps $|\tilde{\Phi}\rangle\langle\tilde{\Phi}|_{EA}$ to a density operator on EA' that can be realized by an ensemble of pure states $\{p_a, |\Psi_a\rangle_{EA'}\} \equiv \{|\tilde{\Psi}_a\rangle_{EA'}\}$:

$$\boxed{I \otimes \mathcal{E}\left(|\tilde{\Phi}\rangle_{EA}\langle\tilde{\Phi}|_{EA}\right) = \sum_a |\tilde{\Psi}_a\rangle_{EA'}\langle\tilde{\Psi}_a|_{EA'} .}$$

We use $|\tilde{\Psi}_a\rangle_{EA'} = \sqrt{p_a}\,|\Psi_a\rangle_{EA}$ as in previous lectures.

The state on the RHS specifies $\mathcal{E}$ completely.

Now we use that

$$|\varphi\rangle_A = \sum_i \varphi_i |i\rangle_A = \sum_i \varphi_i \, \langle i|_E \tilde{\Phi}\rangle_{EA} =$$

$$= \langle \varphi^*|_E \tilde{\Phi}\rangle_{EA}$$

[This is similar to the transpose trick (W.3.7.12):

$$(M_A \otimes I_B)|\tilde{\Phi}\rangle_{AB} = (I_A \otimes M_B^T)|\tilde{\Phi}\rangle_{AB}$$

$$M_A = \sum_{i,j} M_{ij} |i\rangle_A\langle j|_A \, , \quad M_B^T = \sum_{i,j} M_{ji} |i\rangle_B \langle j|_B$$

Proof:

$$\sum_\mu \sum_{i,j} M_{ij} |i\rangle_A\langle j|_A |\mu\rangle_A |\mu\rangle_B = \sum_{i,j} M_{ij} |i\rangle_A |j\rangle_B$$

$$= \sum_{i,j} (M^T)_{ji} |i\rangle_A |j\rangle_B = \sum_\mu \sum_{i,j} (M^T)_{ji} |j\rangle_B\langle i|_B |\mu\rangle_A |\mu\rangle_B$$]

The linearity of the quantum channel $\mathcal{E}$, then yields

$$\mathcal{E}(|\varphi\rangle_A\langle\varphi|_A) = \mathcal{E}\left(\langle\varphi^*|_E \tilde{\Phi}\rangle_{EA} \langle \tilde{\Phi}|_{EA} \varphi^*\rangle_E \right) =$$

$$= \sum_a \langle \varphi^*|_E \Psi_a\rangle_{EA} \langle \Psi_a|_{EA} \varphi^*\rangle_E$$

State lies in A'

This scheme of extracting the action on $|\varphi\rangle_A$ using the dual vector $\langle\varphi^*|_E$ is called the "relative-state method".

Given a state $|\Psi_a\rangle_{EA'}$, when H_E has the same dimension as H_A (let's call it d), we can define an operator M_a,

$M_a : \mathcal{H}_A \longrightarrow \mathcal{H}_{A'}$ by

$$\boxed{M_a|\varphi\rangle_A = \langle\varphi^*|_E \tilde{\Psi}_a\rangle_{EA'} .}$$

Note that this implies

$$M_a \overbrace{\langle\varphi^*|_E \tilde{\Phi}\rangle_{EA}}^{=|\varphi\rangle_A} = \langle\varphi^*|_E \tilde{\Psi}_a\rangle_{EA'} \quad \forall \langle\varphi^*|_E$$

$$\Rightarrow \boxed{(I \otimes M_a)|\tilde{\Phi}\rangle_{EA} = |\tilde{\Psi}_a\rangle_{EA'}}$$

The linear (Kraus) operator M_a is defined via the image state $|\tilde{\Psi}_a\rangle_{EA'}$ it maps the maximally entangled state $|\tilde{\Phi}\rangle_{EA}$ to.

Let's check that M_a is linear:

$$M_a\left(c_1|\varphi\rangle_A + c_2|\xi\rangle_A\right) = c_1\langle\varphi^*|_E\Psi\rangle_{EA'}$$

$$+ c_2\langle\xi^*|_E\Psi\rangle_{EA'} = c_1 M_a|\varphi\rangle_A + c_2 M_a|\xi\rangle_A.$$

Thus, we constructed an operator-sum representation of $\mathcal{E}$:

$$\mathcal{E}\left(|\varphi\rangle_A\langle\varphi|_A\right) = \sum_a \langle\varphi^*|_E\Psi_a\rangle_{EA'}\langle\Psi_a|_{EA'}\varphi^*\rangle_E =$$

$$= \sum_a M_a|\varphi\rangle_A\langle\varphi|_A M_a^\dagger .$$

By convex linearity

$$\mathcal{E}(\rho) = \sum_a M_a\rho M_a^\dagger$$ for every ensemble of states $\{|\tilde{\Psi}_a\rangle\}$.

Choi matrix representation of quantum channel:

Saw that channel $\mathcal{E}$ is fully characterized by Choi matrix

$$(I_E \otimes \mathcal{E}) |\tilde{\Phi}\rangle_{EA} \langle\tilde{\Phi}|_{EA} =$$

$$= \sum_{i,j} |i\rangle_E \langle j|_E \otimes \mathcal{E}(|i\rangle_A \langle j|_A)$$

We also use common notation for that state $\rho_\mathcal{E} = \frac{1}{d} \sum_a |\tilde{\Psi}_a\rangle_{EA} \langle \tilde{\Psi}_a|_{EA}$:

$$\rho_\mathcal{E} = \frac{1}{d} \sum_{i,j} E_{ij} \otimes \mathcal{E}(E_{ij})$$ with $E_{ij} = |i\rangle\langle j|$ is a matrix with 1 in the (i,j)th entry and 0 elsewhere.

- $\mathcal{E}$ is CP iff $\rho_\mathcal{E} \geq 0$ (positive semidefinite)
- $\mathcal{E}$ is TP iff $Tr_B(\rho_\mathcal{E}) = I/d$
- $\mathcal{E}$ is unital iff $Tr_A(\rho_\mathcal{E}) = I/d$
- $\mathcal{E}(\rho) = d\, Tr_A[(\rho^T \otimes I)\rho_\mathcal{E}]$

[action of $\mathcal{E}$ on any state $\rho \in \mathcal{L}(H_d)$]
$d = 2^n$

Summary:

Isomorphism between states $\{|\tilde{\Psi}_a\rangle_{EA'}\}$,

$\rho = \sum_a |\tilde{\Psi}_a\rangle_{EA'}\langle\tilde{\Psi}_a|_{EA'}$ and CP maps

$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger$.

(1) CP map from $\mathcal{H}_A \to \mathcal{H}_{A'}$ $\longrightarrow$ state $\in \mathcal{H}_E \otimes \mathcal{H}_{A'}$

$$(I \otimes \mathcal{E})\left(|\tilde{\Phi}\rangle_{EA}\langle\tilde{\Phi}|_{EA}\right) = \sum_a |\tilde{\Psi}_a\rangle_{EA'}\langle\tilde{\Psi}_a|_{EA'}$$

(2) state $\longrightarrow$ CP map

Ensemble $\{|\tilde{\Psi}_a\rangle\}$ defines $M_a|\varphi\rangle_A = \langle\varphi^*|_E \tilde{\Psi}_a\rangle_{EA'}$

$\Rightarrow$ then use M_a in $\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger$.

If $\mathcal{E}$ is trace preserving, we find $\sum_a M_a^\dagger M_a = I$ directly.

In short: As $\mathcal{E}_{A\to A'}$ is completely positive, $I\otimes\mathcal{E}$ maps a maximally entangled state on EA to a density operator on EA'. This density operator can be expressed as an ensemble of pure states $\{|\tilde{\Psi}_a\rangle\}$ and each of these pure states is associated with a Kraus operator in the operator-sum representation of $\mathcal{E}$.

Freedom of choosing Kraus operators $\{M_a\}$ representing the same channel is identical to choosing different ensembles of pure states $\{|\tilde{\Psi}_a\rangle\}$, $\{|\tilde{\xi}_a\rangle\}$ representing the same density operator (see lecture 2).

$$\Rightarrow (I\otimes\mathcal{E})\,|\tilde{\Phi}\rangle\langle\tilde{\Phi}|_{EA} = \sum_a |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|_{EA'} =$$

$$= \sum_b |\tilde{\xi}_b\rangle\langle\tilde{\xi}_b|_{EA'}$$

where $|\xi_b\rangle = \sum_a V_{ba} |\tilde{\Psi}_a\rangle$ Unitary freedom in the ensemble of given density matrix

unitary (by extending isometry)

Correspondingly: $N_b = \sum_a V_{ba} M_a$ with unitary V_{ba}

represent the same channel $\mathcal{E}$. If $a_{max} \neq b_{max}$, we append zeroes to the shorter list of operators so they have the same size.

Canonical choice of Kraus operators arises from spectral decomposition of density operator:

$$(I \otimes \mathcal{E}) |\tilde{\Phi}\rangle_{EA} \langle \tilde{\Phi}|_{EA} = \sum_i |\tilde{i}\rangle_{EA'} \langle \tilde{i}|_{EA'}$$

such that $\langle \tilde{i} | \tilde{j} \rangle = \delta_{ij}$.

Then the corresponding Kraus operators $\{M_i\}$ are orthogonal with respect to the Hilbert-Schmidt inner product $\mathrm{Tr}(M_i^\dagger M_j) = \mathrm{Tr}(M_i^\dagger M_i)\,\delta_{ij}$.

Follows from

$$\delta_{ij} \langle \tilde{i} | \tilde{i} \rangle = \langle \tilde{i} | \tilde{j} \rangle = \langle \tilde{\Phi} |_{EA} I \otimes M_i^\dagger M_j | \tilde{\Phi} \rangle_{EA} =$$

$$= \sum_{k,l} \langle k|_E \langle k|_A \left(I \otimes M_i^\dagger M_j \right) |l\rangle_E |l\rangle_A =$$

$$= \sum_{l} \langle l|_A M_i^\dagger M_j | l_A \rangle = \mathrm{Tr}\left[M_i^\dagger M_j \right]. \quad \square$$

How many Kraus operators are needed to describe a channel?

One Kraus operator M_a for every state $|\tilde{\Psi}_a\rangle$ in the ensemble $\{|\tilde{\Psi}_a\rangle\}$ of a density operator ρ.
The minimal number of Kraus operators thus corresponds to the rank r of ρ (i.e. the number of its nonzero eigenvalues), which is $r \leq \dim(H_A)\dim(H_{A'})$.
Can choose operator-sum representation with more elements,

of course, just like choosing ensemble $\{|\tilde{\Psi}_a\rangle\}$ with linearly dependent states.

How to construct Kraus operators from Choi matrix $\rho_{\mathcal{E}}$:

Recall $$\rho_{\mathcal{E}} = \frac{1}{d}\sum_{i,j} E_{ij} \otimes \mathcal{E}(E_{ij}) = \sum_a |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|$$

Let's focus on the canonical Kraus representation here, i.e., $\sum_a |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|$ is a spectral decomposition of $\rho_{\mathcal{E}}$ (recall that $|\tilde{\Psi}_a\rangle = \sqrt{\lambda_a}\,|\Psi_a\rangle$ with eigenvalue λ_a and $\langle\Psi_a|\Psi_a\rangle = 1$).

We obtain M_a from
$$M_a|\varphi\rangle_A = {}_E\langle\varphi^*|\tilde{\Psi}_a\rangle_{EA}$$

Choose $|\varphi\rangle_A = |l\rangle_A$ (computational basis state), and we can always expand

unnormalized eigenvector $|\tilde{\Psi}_a\rangle = \sqrt{\lambda_a}\,|\Psi_a\rangle$ in computational basis.

$$|\tilde{\Psi}_a\rangle_{EA} = \sum_{i,j} c_{ij}\,|i\rangle_E|j\rangle_A$$

$$\Rightarrow {}_A\langle k|M_a|l\rangle_A = \sum_{i,j} {}_E\langle l|\,{}_A\langle k|\; c_{ij}\,|i\rangle_E|j\rangle_A \overset{\delta_{li}\delta_{kj}}{=}$$

$$= C_{\ell\kappa} .$$

Note that

$$(I \otimes A) |\tilde{\Phi}\rangle = |A\rangle\rangle_c$$

where

$$|A\rangle\rangle_c = \sum_{i,j} A_{ij} \, |j\rangle \otimes |i\rangle$$

is the column-vectorization of matrix A_{ij}.

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \longrightarrow |A\rangle\rangle_c = \begin{pmatrix} A_{11} \\ A_{21} \\ A_{12} \\ A_{22} \end{pmatrix}$$

In contrast.

$$(A \otimes I) |\tilde{\Phi}\rangle = |A\rangle\rangle_r$$

is the row-vectorization of A

$$|A\rangle\rangle_r = \sum_{i,j} A_{ij} \, |i\rangle \otimes |j\rangle$$

such that

$$|A\rangle\rangle_r = \begin{pmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{pmatrix}$$

In contrast, in the Choi-matrix we