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Stinespring dilation

Explicit construction of unitary map on extended system that realizes $\mathcal{E}$.

Introduced the isometry $\mathcal{U}_{A\to A'E}$ earlier

$$\mathcal{U}_{AE} : |\psi\rangle_A |0\rangle_E \longmapsto \sum_a M_a |\psi\rangle_A |a\rangle_E$$

This isometry yields $\mathcal{E}: A \to A'$ when performing Tr_E.

Completeness relation $\sum_a M_a^\dagger M_a = I \iff \mathcal{U}_{AE}^\dagger \mathcal{U}_{AE} = I_A$.

This isometry $\mathcal{U}_{AE}$ is called the Stinespring dilation of the channel $\mathcal{E}$ defined by its Kraus operators $\{M_a\}$.

Note that $\mathcal{U}_{AE}$ can be completed to a unitary by choosing remaining columns perpendicular to the first one, i.e., by expanding the input Hilbert space.

Alternative way of constructing U_{AE} is by purification of the density operator $\sum_a |\tilde{\Psi}_a\rangle_{RA'}\langle\tilde{\Psi}_a|_{RA'}$ arising from the channel-state duality.

Purification reads ($\dim(E) = a_{max} \equiv$ # Kraus operators):

$$|\Psi\rangle_{RA'E} = \sum_a |\tilde{\Psi}_a\rangle_{RA'} |a\rangle_E .$$

This pure state is obtained from acting with U_{AE} on the maximally entangled state $|\tilde{\Phi}\rangle_{RA} = \sum_i |i\rangle_R |i\rangle_A$:

$$(I_R \otimes U_{AE}) |\tilde{\Phi}\rangle_{RA} |0\rangle_E = \sum_a M_a |\tilde{\Phi}\rangle_{RA} |a\rangle_E =$$

$$= \sum_a |\tilde{\Psi}_a\rangle_{RA'} |a\rangle_E = |\Psi\rangle_{RA'E} .$$

see lecture 5:

$$M_a |\varphi\rangle_A = \langle\varphi^*|_R \tilde{\Psi}_a\rangle_{RA'} \quad \Leftrightarrow \quad M_a \langle\varphi^*|_R \tilde{\Phi}\rangle_{RA} = \langle\varphi^*|_R \tilde{\Psi}_a\rangle_{RA'}$$

$$\Rightarrow (I_R \otimes M_a) |\tilde{\Phi}\rangle_{RA} = |\tilde{\Psi}_a\rangle_{RA'}$$

We may find the Stinespring dilation $\mathcal{U}_{AE}$ from the pure state $|\bar{\Psi}\rangle_{RA'E}$ via

$$\mathcal{U}_{AE}|\psi\rangle_A = \langle\psi^*|_R \, \bar{\Psi}\rangle_{RA'E} .$$

Show this explicitly

$$\langle\psi^*|_R \bar{\Psi}\rangle_{RA'E} = \langle\psi^*|_R \sum_a M_a |\tilde{\Phi}\rangle_{RA} |a\rangle_E =$$

$$= \sum_a M_a |\psi\rangle_A |a\rangle_E = \mathcal{U}_{AE}|\psi\rangle_A . \quad \square$$

$$\langle\psi^*|_R \tilde{\Phi}\rangle_{RA} = |\psi\rangle_A$$

Summary: Can characterize a quantum channel $\mathcal{E}: A \to A'$ as a pure state $|\bar{\Psi}\rangle_{RA'E}$. This state arises from purification of density operator $\sum_a |\tilde{\Psi}_a\rangle\langle\tilde{\Psi}_a|$.

Examples of quantum channels:

Focus on realistic single qubit channels:

1. Depolarizing channel (symmetric Pauli channel)
2. Dephasing channel
3. Amplitude damping channel

Also discuss the various channel representations & how to move between them

- Wood paper, Forest benchmarking docs, Greenbaum GST

(i) **Stinespring isometry** representation $U_{A\to AE}$

(ii) **Kraus = operator-sum** representation: $\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger$

(iii) **Choi matrix** $\rho_\mathcal{E} = \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$, $d = 2^n$, $n = \#$ qubits

(iv) **Process matrix** = chi matrix = χ matrix

$$\mathcal{E}(\rho) = \sum_{i,j} \chi_{ij} P_i \rho P_j^\dagger$$

||| (iii) when choosing col-vec basis in (iv)

$\hat{=}$ Choi matrix when going from Pauli to col-vec basis E_{ji}

general orthonormal operator basis, here Pauli basis $P_i^{\otimes n}$

$\Rightarrow$ Kraus operators are eigenvectors of Choi matrix & also process matrix

Here, we choose the Pauli basis $P_i^{\otimes n}$, $P_0 = I$, $P_1 = X$, $P_2 = Y$, $P_3 = Z$.

Note, when choosing the col-vec basis $P_\alpha = E_{ji} = |j\rangle\langle i|$ with $\alpha = i + dj$ $(d = 2^n)$, then $\chi_{ij} \equiv \rho_\mathcal{E}$ i.e., the process matrix $\equiv$ Choi matrix.

(V) Liouville superoperator (here in col-vec basis)

Relies on vectorization of density matrix $\rho \mapsto |\rho\rangle\rangle_\sigma$.

Different basis choices are possible — orthonormal operator basis

- Pauli basis: then the Liouville superoperator $\equiv$ Pauli transfer matrix

- col-vec

$$|A\rangle\rangle_c = \sum_{i,j=1}^{d} A_{ij} |j\rangle \otimes i \quad \text{(for the density matrix)}$$

Then, $|\mathcal{E}(\rho)\rangle\rangle_c = \mathcal{L}_\mathcal{E} |\rho\rangle\rangle_c = \sum_{\mu,m} \mathcal{L}_{nm,\nu\mu} \rho_{\mu\nu}$

Matrix map: $|\rho\rangle\rangle_c \xmapsto{\mathcal{L}_\mathcal{E}} |\mathcal{E}(\rho)\rangle\rangle_c$

- row-vec

$$|A\rangle\rangle_r = \sum_{i,j} A_{ij} |i\rangle \otimes |j\rangle$$

(vi) Pauli transfer matrix $(R_{\mathcal{E}})_{ij} = \frac{1}{d} \mathrm{Tr}\left[P_i \, \mathcal{E}(P_j)\right]$
(superoperator)

$d = 2^n$, $n = \#$qubits

Superoperator in Pauli basis

Some useful properties:

- Choi matrix $\rho_{\mathcal{E}} \geq 0$ (positive semidefinite) iff $\mathcal{E}$ is CP. Follows as $\rho_{\mathcal{E}}$ is the image of $|\Phi\rangle\langle\Phi|$ under $\mathcal{E}$.

- If we define $P_0 = I_d = II \cdots I$, then $\mathcal{E}$ is TP iff first row of $R_{\mathcal{E}}$ is vector $(1, 0, 0, \ldots, 0)$

- $\mathcal{E}$ is unital iff first column of $R_{\mathcal{E}}$ is $(1, 0, 0, \ldots, 0)^T$.

- linear tf between Choi matrix $\rho_{\mathcal{E}}$ and PTM $R_{\mathcal{E}}$

$$(R_{\mathcal{E}})_{ij} = \mathrm{Tr}\left[\rho_{\mathcal{E}} \, P_j^T \otimes P_i\right]$$

$$\rho_{\mathcal{E}} = \frac{1}{d^2} \sum_{ij} (R_{\mathcal{E}})_{ij} \, P_j^T \otimes P_i$$

- PTM $(R_{\mathcal{E}})_{ij}$ is a superoperator $\Rightarrow$ composition $\mathcal{E}_2 \circ \mathcal{E}_1$ becomes matrix multiplication

$$\Rightarrow (R_{\mathcal{E}_2 \circ \mathcal{E}_1})_{ij} = \sum_k (R_{\mathcal{E}_2})_{ik} \, (R_{\mathcal{E}_1})_{kj}$$

- When vectorizing ρ in Pauli basis

$$(|\rho\rangle\rangle_P)_i = \mathrm{Tr}(P_i \rho) \Rightarrow \rho = \overbrace{\frac{1}{d} \sum_i \langle\langle P | \rho \rangle\rangle_P \, |P\rangle\rangle}^{= \frac{1}{d} \vec{\rho} \cdot \vec{P}}$$

$$\mathcal{E}(\rho) \hat{=} |\rho'\rangle\rangle_P = R_{\mathcal{E}} |\rho\rangle\rangle_P \, .$$

- Choi matrix is <u>not</u> a superoperator. Becomes Liouville superoperator $\mathcal{L}_{\mathcal{E}}$ under column reshuffling:

$$(\rho_{\mathcal{E}})_{mn,\mu\nu} = (\mathcal{L}_{\mathcal{E}})_{\nu n, \mu m}$$

Evolution of state described by

$$\mathcal{E}(\rho) = \mathrm{Tr}_A \left[(\rho^T \otimes I_E) \rho_{\mathcal{E}} \right]$$

- Roth's lemma: $|ABC\rangle\rangle_c = (C^T \otimes A) |B\rangle\rangle_c$

$\Rightarrow \mathcal{L}_{\mathcal{E}} = \sum_a M_a^* \otimes M_a$

① Depolarizing channel

- Qubit remains intact with probability $1-p$
- Error occurs with probability p

Three possible Pauli errors $\{X, Y, Z\}$, all with same prob. p

- Bit flip X: $|0\rangle \mapsto |1\rangle$, $|1\rangle \mapsto |0\rangle$
- Phase flip Z: $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto -|1\rangle$
- Both $ZX = Y$: $|0\rangle \mapsto i|1\rangle$, $|1\rangle \mapsto -i|0\rangle$

(i) Stinespring isometry (unitary representation)

$$\mathcal{U}_{A\to AE}: |\psi\rangle_A \mapsto \sqrt{1-p}\, |\psi\rangle_A |0\rangle_E + \sqrt{\frac{p}{3}} \left[X|\psi\rangle_A |1\rangle_E + Y|\psi\rangle_A |2\rangle_E + Z|\psi\rangle_A |3\rangle_E \right]$$

If we would measure E in the basis $|a\rangle_E$, $a = 0,1,2,3$, we would know which error occurred and could reverse it.

(ii) Kraus or operator-sum representation

Perform $\mathrm{Tr}_E(U_{A\to AE})$: $M_a = {}_E\langle a| U_{A\to AE} |0\rangle_E$.

$\Rightarrow M_0 = \sqrt{1-p}\, I$, $M_1 = \sqrt{\frac{p}{3}}\, X$, $M_2 = \sqrt{\frac{p}{3}}\, Y$,

$M_3 = \sqrt{\frac{p}{3}}\, Z$

Check completeness : $\sum_a M_a^\dagger M_a = \left[1-p+3\cdot\frac{p}{3}\right] I = I$

Channel evolution:

$$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger =$$

$$= (1-p)\rho + \frac{p}{3}\left[X\rho X + Y\rho Y + Z\rho Z\right]$$

(iii) Choi matrix (relative state) $\hat{=}$ the density matrix that $(I\otimes\mathcal{E})$ maps the maximally entangled state to

$$\underbrace{(I\otimes\mathcal{E})\left(|\tilde{\Phi}\rangle\langle\tilde{\Phi}|\right)}_{} = \sum_a |\Psi_a\rangle\langle\Psi_a| = \rho_{\mathcal{E}}$$

$$= \rho_{\mathcal{E}} = \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$$

Here:

$$|\tilde{\Phi}\rangle\langle\tilde{\Phi}| = \underbrace{(|00\rangle+|11\rangle)(\langle 00|+\langle 11|)}_{=|\tilde{\Phi}^+\rangle\langle\tilde{\Phi}^+| =}$$

Four Bell basis states

$$\Rightarrow \rho_E = (1-p)\,|\tilde{\Phi}\rangle\langle\tilde{\Phi}| + \frac{p}{3}\Big[\overbrace{(|01\rangle+|10\rangle)(\langle 01|+\langle 10|)}^{=|\tilde{\Psi}^+\rangle\langle\tilde{\Psi}^+|}$$

$$+ \overbrace{(|01\rangle-|10\rangle)(\langle 01|-\langle 10|)}^{=|\tilde{\Psi}^-\rangle\langle\tilde{\Psi}^-|}$$

$$+ \underbrace{(|00\rangle-|11\rangle)(\langle 00|-\langle 11|)}_{=|\tilde{\Phi}^-\rangle\langle\tilde{\Phi}^-|}\Big] =$$

$$= (1-p)\,|\tilde{\Phi}^+\rangle\langle\tilde{\Phi}^+| + \frac{p}{3}\Big[|\tilde{\Psi}^+\rangle\langle\tilde{\Psi}^+| + |\tilde{\Psi}^-\rangle\langle\tilde{\Psi}^-| + |\tilde{\Phi}^-\rangle\langle\tilde{\Phi}^-|\Big].$$

For $p = \frac{3}{4}$, we find $\rho_E = \frac{I}{2}$, i.e., the maximally entangled state evolves into the maximally mixed density matrix.

this is not $I/4$ due to $|\tilde{\Phi}\rangle\langle\tilde{\Phi}|$ basis norm $\sqrt{d}$.

The action on system A (the second qubit) then becomes

$$\mathcal{E}_{p=3/4}(|\psi\rangle\langle\psi|) = \langle\psi^*|_R \frac{I}{2} |\psi^*\rangle_R = \frac{I}{2}.$$

(relative-state method)

At $p = 3/4$, the qubit is mapped to $\frac{I}{2}$, irrespective of initial state $|\psi\rangle$. Thus, we can also write the Choi-matrix of the depolarizing channel as

$$\rho_\mathcal{E} = \left[\left(1 - \frac{4}{3}p\right) |\tilde{\Phi}\rangle\langle\tilde{\Phi}| + \frac{4}{3}p \frac{I_{EA}}{2} \right] \frac{1}{2}$$

to have $\mathrm{Tr}\,\rho_\mathcal{E} = 1$

→ sometimes one thus defines $\mathcal{E}(\rho) = (1 - \frac{3}{4}p')\rho + \frac{p'}{4}(X\rho X + Y\rho Y + Z\rho Z)$ with $p' = \frac{4}{3}p$.

Explicitly:

$$\rho_\mathcal{E} = \frac{1}{2} \begin{pmatrix} 1 - \frac{2}{3}p & 0 & 0 & 1 - \frac{4}{3}p \\ 0 & \frac{2}{3}p & 0 & 0 \\ 0 & 0 & \frac{2}{3}p & 0 \\ 1 - \frac{4}{3}p & 0 & 0 & 1 - \frac{2}{3}p \end{pmatrix}$$

Obtain canonical Kraus operators M_a from $\rho_\mathcal{E}$:

Diagonalize:

$\lambda_0 = 1 - p$, $v_0 = (1, 0, 0, 1)^T \cdot \frac{1}{\sqrt{2}}$

$\lambda_1 = \frac{p}{3}$, $v_1 = (-1, 0, 0, 1)^T \cdot \frac{1}{\sqrt{2}}$

$\lambda_2 = \frac{p}{3}$, $v_2 = (0, 0, 1, 0)^T$.

$\lambda_3 = \frac{p}{3}$, $v_3 = (0, 1, 0, 0)^T$

Kraus operators follow from

$$M_a = \sqrt{\lambda_a}\ \mathrm{unvec}\left(|v_a\rangle\rangle\right)$$

$$\Rightarrow\ M_0 = \sqrt{1-p}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad M_1 = \sqrt{\frac{p}{3}}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sqrt{\frac{p}{3}}\frac{Z}{\sqrt{2}}$$

$$M_2 = \sqrt{\frac{p}{3}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sqrt{\frac{p}{3}}\,\sigma^+ = \sqrt{\frac{p}{3}}\,\frac{1}{2}(X+iY)$$

$$M_3 = \sqrt{\frac{p}{3}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sqrt{\frac{p}{3}}\,\sigma^- = \sqrt{\frac{p}{3}}\,\frac{1}{2}(X-iY)$$

$$\sum_a M_a^+ M_a = \frac{1-p}{2}I + \frac{p}{6}I + \frac{p}{3}I = \frac{I}{2}.$$

$\Rightarrow$ multiply all M_a with $\sqrt{2}$ to ensure $\sum_a M_a^+ M_a = I$.

$$\Rightarrow\ M_0 = \sqrt{1-p}\,I,\quad M_1 = -\sqrt{\frac{p}{3}}\,Z,\quad M_2 = \sqrt{\frac{p}{3}}\sqrt{2}\,\sigma^+$$

$$M_3 = \sqrt{\frac{p}{3}}\sqrt{2}\,\sigma^-$$

Unitary tf to original Kraus operators

$$N_0 = \sqrt{1-p}\,I,\quad N_1 = \sqrt{\frac{p}{3}}\,X,\quad N_2 = \sqrt{\frac{p}{3}}\,Y,\quad N_3 = \sqrt{\frac{p}{3}}\,Z.$$

$$N_a = \sum_b V_{ab} M_b,\quad V_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -i/\sqrt{2} & i/\sqrt{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\sqrt{\tfrac{p}{3}}X \equiv N_1 = (M_2+M_3)/\sqrt{2}$$

$$\sqrt{\tfrac{p}{3}}Y = N_2 = -\frac{i}{\sqrt{2}}(M_2-M_3)$$

$$\Downarrow$$

$$V^\dagger V = I$$

is indeed unitary ✓.

(iv) Process χ matrix (in Pauli basis) $\hat{=}$ Choi matrix in Pauli basis

Expand $M_a = \sum_b C_{ab} P_b$ in Pauli basis:

$P_a \in \{I, X, Y, Z\}^{\otimes n}$. Here, $n=1$.

$\Rightarrow M_0 = \sqrt{1-p}\, I,\ M_1 = \sqrt{\tfrac{p}{3}}\, X,\ M_2 = \sqrt{\tfrac{p}{3}}\, Y,\ M_3 = \sqrt{\tfrac{p}{3}}\, Z$

$\Rightarrow$ Kraus operators for depolarizing channel already diagonal in the Pauli basis:

$$\mathcal{E}(\rho) = \sum_{i,j} \chi_{ij} P_i \rho P_j^\dagger$$

$$\Rightarrow \chi_{ij} = \begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & \frac{p}{3} & 0 & 0 \\ 0 & 0 & \frac{p}{3} & 0 \\ 0 & 0 & 0 & \frac{p}{3} \end{pmatrix}$$

$\chi^\dagger = \chi$, $\chi \geq 0$. Mapping to a completely mixed state at $p = \frac{3}{4}$.

$\Rightarrow \chi_{ij}(p = 3/4) = \frac{1}{4} I$.

(v) Louiville superoperator in col-vec basis

$$\mathcal{L}_\mathcal{E} = \sum_a M_a^* \otimes M_a =$$

$$= (1-p)\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{I\otimes I} + \frac{p}{3}\underbrace{\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}}_{X\otimes X}$$

$$+ \frac{p}{3}\underbrace{\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}}_{Y^*\otimes Y} + \frac{p}{3}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{2}{3}p & 0 & 0 & \frac{2}{3}p \\ 0 & 1-\frac{4}{3}p & 0 & 0 \\ 0 & 0 & 1-\frac{4}{3}p & 0 \\ \frac{2}{3}p & 0 & 0 & 1-\frac{2}{3}p \end{pmatrix}$$

$$\mathcal{L}_\mathcal{E} |\rho\rangle\rangle_c = |\rho'\rangle\rangle_c .$$

Is related to $\rho_\mathcal{E}$ by column reshuffling operation:

$$R: Q_{m,\mu;n,\nu} \longmapsto Q_{\nu,\mu;n,m} \qquad \underset{\Downarrow}{H_1} \qquad \underset{\Downarrow}{H_2}$$

$$Q_{m,\mu;n,\nu} = \langle m,\mu | Q | n,\nu\rangle \quad , \quad |n,\nu\rangle \equiv |n\rangle \otimes |\nu\rangle$$

vi) Pauli transfer matrix (PTM) or Pauli-Liouville representation:

$$(R_\mathcal{E})_{ij} = \frac{1}{d} \mathrm{Tr}\left[P_i \, \mathcal{E}(P_j)\right]$$

$$P_i \in \{I, X, Y, Z\}^{\otimes n}$$

Here: $\mathcal{E}(A) = (1-p)A + \frac{p}{3}\left[XAX + YAY + ZAZ\right].$

Pauli's either commute or anti-commute.

$(1\,0\,0\,0) \hat{=}$ TP channel

$$(R_\mathcal{E}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-\frac{4}{3}p) & 0 & 0 \\ 0 & 0 & (1-\frac{4}{3}p) & 0 \\ 0 & 0 & 0 & (1-\frac{4}{3}p) \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \hat{=}$ unital channel

$\mathcal{E}(I) = I$; $\mathcal{E}(X) = (1-\frac{4}{3}p)X$, $\mathcal{E}(Y) = (1-\frac{4}{3}p)Y$

General $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} \Rightarrow |\rho\rangle\rangle_\sigma = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix}.$

$$\Rightarrow \mathcal{E}(\rho) = (R_\mathcal{E}) \cdot |\rho\rangle\rangle_\sigma = \begin{pmatrix} 1 & & & \\ & 1-\frac{4}{3}p & & \\ & & 1-\frac{4}{3}p & \\ & & & 1-\frac{4}{3}p \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ r_x\left(1-\frac{4}{3}p\right) \\ r_y\left(1-\frac{4}{3}p\right) \\ r_z\left(1-\frac{4}{3}p\right) \end{pmatrix} = \frac{1}{2}\left(I + \left(1-\frac{4}{3}p\right)\vec{r}\cdot\vec{\sigma}\right) =$$

$$\Leftrightarrow \mathcal{E}(\rho) = \frac{1}{2}\left[I + \vec{r}\,'\cdot\vec{\sigma}\right]$$

with $\vec{r}\,' = \left(1-\frac{4}{3}p\right)\vec{r}$ (shrinking of all components of $\vec{r}$).

Note:

One can find $\rho_{\mathcal{E}}$ (Choi matrix in col-vec basis) from the PTM $(R_{\mathcal{E}})_{ij}$ as

$$\rho_{\mathcal{E}} = \frac{1}{d^2}\sum_{i,j=1}^{d^2} (R_{\mathcal{E}})_{ij}\; P_j^{+} \otimes P_i$$

The PTM is diagonal for the depolarizing channel. Thus,

$$P_0^T \otimes P_0 = I_4\,,\quad P_1^T \otimes P_1 = X^T \otimes X = X \otimes X = \begin{pmatrix} 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \end{pmatrix}$$

$$P_2^T \otimes P_2 = Y^T \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \end{pmatrix}$$

$$P_3^T \otimes P_3 = z^T \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

The Choi (CJ) matrix thus reads

$$\rho_\varepsilon = \frac{1}{4}\left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \left(1-\tfrac{4}{3}p\right) \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \right.$$

$$\left. + \left(1-\tfrac{4}{3}p\right) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \right] =$$

$$= \frac{1}{4} \begin{pmatrix} 2-\frac{4}{3}p & 0 & 0 & 2-\frac{8}{3}p \\ 0 & \frac{4}{3}p & 0 & 0 \\ 0 & 0 & \frac{4}{3}p & 0 \\ 2-\frac{8}{3}p & 0 & 0 & 2-\frac{4}{3}p \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1-\frac{2}{3}p & 0 & 0 & 1-\frac{4}{3}p \\ 0 & \frac{2}{3}p & 0 & 0 \\ 0 & 0 & \frac{2}{3}p & 0 \\ 1-\frac{4}{3}p & 0 & 0 & 1-\frac{2}{3}p \end{pmatrix}$$

This agrees with our earlier result.

Graphical representation of depolarizing channel:

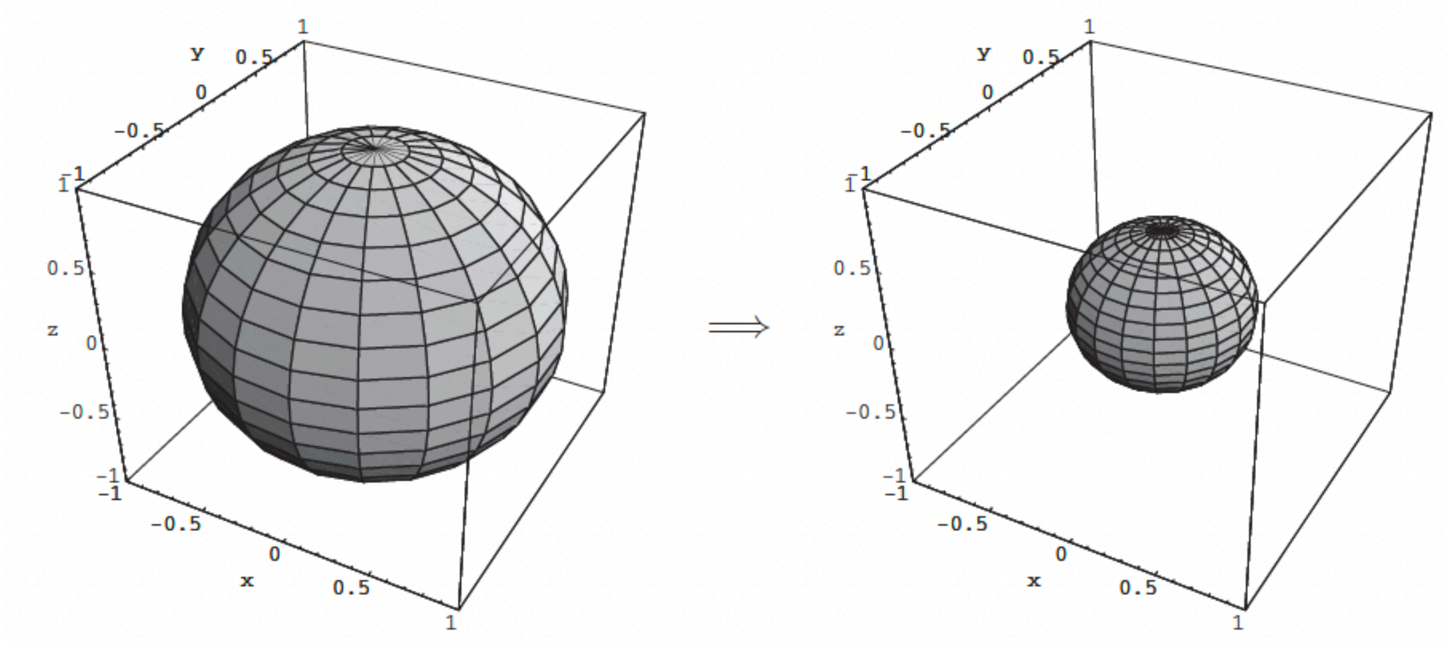

Figure 8.11. The effect of the depolarizing channel on the Bloch sphere, for $p = 0.5$. Note how the entire sphere contracts uniformly as a function of p.

From Nielsen & Chuang

(B) Dephasing channel (or phase damping channel):

(i) Unitary representation (Stinespring isometry):

$$|0\rangle_A |0\rangle_E \longmapsto \sqrt{1-p}\, |0\rangle_A |0\rangle_E + \sqrt{p}\, |0\rangle_A |1\rangle_E$$

$$|1\rangle_A |0\rangle_E \longmapsto \sqrt{1-p}\, |1\rangle_A |0\rangle_E + \sqrt{p}\, |1\rangle_A |2\rangle_E$$

Qubit does not make a transition, but the environment E „scatters" off the qubit with probability p and makes a transition into state $|1\rangle_E$ or $|2\rangle_E$, depending on the state of the qubit.

The channel picks a preferred basis for qubit A, the computational basis $\{|0\rangle_A, |1\rangle_A\}$ is the only one where no bit flips of A occur. Explicitly,

$$|+\rangle_A = \frac{1}{\sqrt{2}}\left[|0\rangle_A + |1\rangle_A\right]|0\rangle_E \longmapsto \frac{1}{\sqrt{2}}\Big[\sqrt{1-p}\,\big(|0,0\rangle + |1,0\rangle\big)$$

$$+\sqrt{p}\,\big(|0,1\rangle + |1,2\rangle\big)\Big] =$$

Bit flips occur in any basis other than the Z basis

$$= \sqrt{1-p}\, |+\rangle_A |0\rangle_E + \sqrt{p}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}|1\rangle_E + \frac{|+\rangle - |-\rangle}{\sqrt{2}}|2\rangle_E\right)$$

(ii) Kraus operators :

$$M_a = \langle a|_E \, U_{AE} \, |0\rangle_E$$

$$\Rightarrow \quad M_0 = \sqrt{1-p}\, I \quad \rightarrow \quad \mathcal{E}(\rho) = \sum_{a=1}^{3} M_a \rho M_a^{\dagger}$$

$$M_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$$

Check: $M_0^{\dagger} M_0 + M_1^{\dagger} M_1 + M_2^{\dagger} M_2 = (1-p) I + p I = I$ ✓

We can find an isometry to a Kraus representation with only 2 Kraus operators.

This can be done easily when going from ρ_E to M_a.

(iii) Choi matrix $\rho_\mathcal{E}$ (in col-vec):

$$\rho_\mathcal{E} = (I \otimes \mathcal{E}) |\tilde{\Phi}\rangle\langle\tilde{\Phi}| =$$

$$= \sum_{i,j} |i\rangle_E\langle j|_E \otimes \mathcal{E}(|i\rangle_A\langle j|_A) =$$

$$= |0\rangle\langle 0| \otimes (1-p)|0\rangle\langle 0| + |0\rangle\langle 0| \otimes p|0\rangle\langle 0|$$

$$+ |0\rangle\langle 1| \otimes (1-p)|0\rangle\langle 1| + |1\rangle\langle 0| \otimes (1-p)|1\rangle\langle 0|$$

$$+ |1\rangle\langle 1| \otimes (1-p)|1\rangle\langle 1| + |1\rangle\langle 1| \otimes p\,|1\rangle\langle p|$$

use M_a to compute $\mathcal{E}(|i\rangle\langle j|)$

Calculational details

$$M_1 |0\rangle\langle 1| M_1^\dagger = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$M_2 |0\rangle\langle 1| M_2^\dagger = \begin{pmatrix} 0 & \\ & \sqrt{p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \\ & \sqrt{p} \end{pmatrix} = \begin{pmatrix} 0 & \\ & \sqrt{p} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \rho_\mathcal{E} = \begin{array}{c} \\ \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle_{EA} \end{array} \\ \begin{pmatrix} 1 & & & 1-p \\ & & 0 & \\ & & & \\ 1-p & & & 1 \end{pmatrix} \end{array} \cdot \frac{1}{2}$$

to have $\mathrm{Tr}\, \rho_\mathcal{E} = 1$.

Find canonical set of Kraus operators $\{M_a\}$:

Diagonalize $\rho_\mathcal{E}$:

$$\lambda_1 = \frac{p}{2}, \quad v_1 = \frac{1}{\sqrt{2}}(-1, 0, 0, 1)^T$$

$$\lambda_2 = 1 - \frac{p}{2}, \quad v_2 = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$$

$$\lambda_3 = 0, \quad v_3 = (0, 1, 0, 0)^T$$

$$\lambda_4 = 0, \quad v_4 = (0, 0, 1, 0)^T$$

$$\Rightarrow M_a = \sqrt{\lambda_a}\ \mathrm{unvec}(|v_a\rangle\rangle)$$

$$\Rightarrow M_0 = \sqrt{1-\frac{p}{2}}\ \frac{c}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \sqrt{\frac{p}{2}}\ \frac{c}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

To have $M_0^\dagger M_0 + M_1^\dagger M_1 = I = \left[\left(1-\frac{p}{2}\right)\cdot\frac{1}{2} + \frac{p}{2}\frac{1}{2}\right]c^2$

$$\Leftrightarrow \frac{c^2}{2} I \overset{!}{=} I \Rightarrow c = \sqrt{2}.$$

$\Rightarrow$ also multiply M_1 by (-1) to bring to "standard form":

$$\boxed{M_0 = \sqrt{1-\frac{p}{2}}\ I, \quad M_1 = \sqrt{\frac{p}{2}}\ Z}$$

Thus:

$$\mathcal{E}(\rho) = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger =$$

$$= \left(1-\frac{p}{2}\right)\rho + \frac{p}{2} Z \rho Z =$$

Now $Z\rho Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} =$

$$= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} & -\rho_{01} \\ \rho_{10} & -\rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{pmatrix}$$

$$\Rightarrow \mathcal{E}(\rho) = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}$$

$$\Rightarrow \underbrace{[\mathcal{E}\circ\mathcal{E}\circ\cdots\circ\mathcal{E}]}_{n \text{ times}}(\rho) = \begin{pmatrix} \rho_{00} & (1-p)^n \rho_{01} \\ (1-p)^n \rho_{10} & \rho_{11} \end{pmatrix} \xrightarrow[p=\frac{\Gamma t}{n}]{n\to\infty} \begin{pmatrix} \rho_{00} & e^{-\Gamma t}\rho_{01} \\ e^{-\Gamma t}\rho_{10} & \rho_{11} \end{pmatrix}$$

Off-diagonal elements (coherences) decay exponentially in time. Γ = probability of Z error ($\hat{=}$ scattering event) per unit time. $\Gamma = \frac{1}{T_2}$ dephasing timescale.

$\rightarrow$ Qubit localizes in (preferred) Z-basis (decoheres).

(iv) Process χ matrix in Pauli basis

$\{I, X, Y, Z\}^{\otimes n}$

Expand Kraus operators in Pauli basis $M_a = \sum_b c_{ab} P_b$

$$\mathcal{E}(\rho) = \sum_{i,j} \chi_{ij} P_i \rho P_j^\dagger$$

$$\Rightarrow \chi_{ij} = \begin{pmatrix} 1-\frac{p}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{p}{2} \end{pmatrix}$$

(v) Louiville superoperator (in col-vec basis)

$$\mathcal{L}_\mathcal{E} = \sum_a M_a^* \otimes M_a = \left(1-\frac{p}{2}\right) I \otimes I + \frac{p}{2} Z \otimes Z =$$

$$= \left(1-\frac{p}{2}\right) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \frac{p}{2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & & & \\ & 1-p & & \\ & & 1-p & \\ & & & 1 \end{pmatrix}$$

(vi) Pauli transfer matrix (Pauli-Liouville representation)

$$(R_\mathcal{E})_{ij} = \frac{1}{d} \mathrm{Tr}\left[P_i \, \mathcal{E}(P_j)\right]$$

$$\mathcal{E}(\rho) = \left(1-\frac{p}{2}\right)\rho + \frac{p}{2} Z \rho Z$$

$$\Rightarrow (R_\mathcal{E}) = \begin{pmatrix} 1 & & & \\ & 1-p & & \\ & & 1-p & \\ & & & 1 \end{pmatrix}$$

Action on single qubit state

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{P}\right) \Rightarrow |\rho\rangle\rangle_P = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix}$$

$$\Rightarrow |\rho'\rangle\rangle_P = \sum_j (R_\mathcal{E})_{ij} \left(|\rho\rangle\rangle_P\right)_j =$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ (1-p)\, r_x \\ (1-p)\, r_y \\ r_z \end{pmatrix} = \frac{1}{2}\left(I + \vec{r}\,'\cdot\vec{P}\right)$$

with $\vec{r}\,' = \begin{pmatrix} (1-p)\, r_x \\ (1-p)\, r_y \\ r_z \end{pmatrix}$

Bloch ball shrinks to a prolate spheroid aligned with z-axis. Under many applications of the channel the ball deflates in the xy-plane, degenerating to the z-axis:

$$\vec{r}\,' \xrightarrow[\text{dephasing}]{\text{continuous}} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

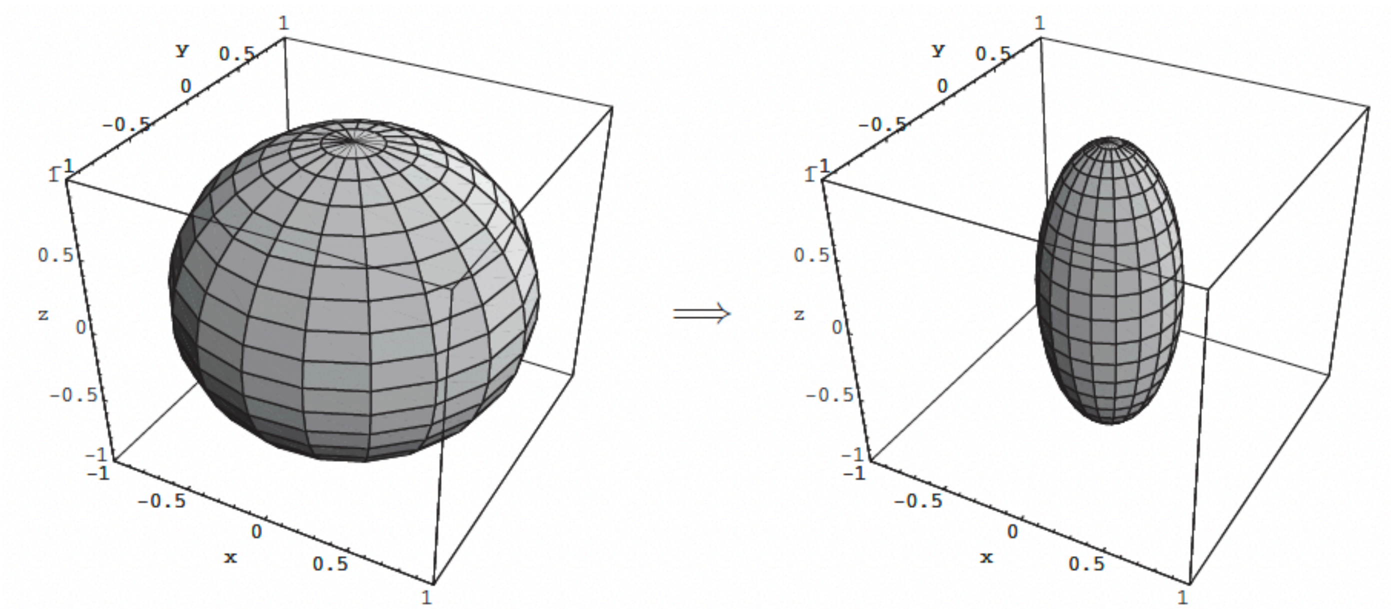

Figure 8.9. The effect of the phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{z}$ axis are left alone, while the $\hat{x} - \hat{y}$ plane is uniformly contracted by a factor of $1 - 2p$.

From N & C.

Amplitude damping channel:

Describes decay processes from excited state (of qubit) to the ground state. Emission of „photon" into the environment.

Associated timescale T_1.

(i) Unitary representation (Stinespring isometry):

no transition as E is in its GS

$$|0\rangle_A |0\rangle_E \longmapsto |0\rangle_A |0\rangle_E$$

$$|1\rangle_A |0\rangle_E \longmapsto \sqrt{1-p}\, |1\rangle_A |0\rangle_E + \sqrt{p}\, |0\rangle_A |1\rangle_E$$

photon in E

(ii) Kraus representation:

$$M_a = \langle a|_E \, U_{AE} |0\rangle_E$$

$$\Rightarrow M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

No jump occurs

Quantum jump from $|1\rangle_A$ to $|0\rangle_A$

$$\mathcal{E}(\rho) = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger =$$

$$= \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} \begin{pmatrix} \rho_{11} & \sqrt{1-p}\,\rho_{12} \\ \rho_{21} & \sqrt{1-p}\,\rho_{22} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p}\,\rho_{12} & 0 \\ \sqrt{p}\,\rho_{22} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{11} & \sqrt{1-p}\,\rho_{12} \\ \sqrt{1-p}\,\rho_{21} & (1-p)\,\rho_{22} \end{pmatrix} + \begin{pmatrix} p\,\rho_{22} & 0 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \rho_{11} + p\,\rho_{22} & \sqrt{1-p}\,\rho_{12} \\ \sqrt{1-p}\,\rho_{21} & (1-p)\,\rho_{22} \end{pmatrix}$$

11 element when $\mathcal{E}$ is applied again

$\rho_{11} + [p + p(1-p)]\rho_{22} =$

$= \rho_{11} + p[1+(1-p)]\rho_{22}$

Again, setting $p = \dfrac{\Gamma t}{n}$ with $\Gamma = \dfrac{1}{T}$ the probability for a decay to occur per unit time $\Delta t = \dfrac{t}{n}$.

Applying the channel n times yields $p = \Gamma \Delta t = \dfrac{\Gamma t}{n} \ll 1$:

$$\mathcal{E}^n(\rho) = \begin{pmatrix} \rho_{11} + [1-(1-p)^n]\,\rho_{22} & (1-p)^{n/2}\,\rho_{12} \\ (1-p)^{n/2}\,\rho_{21} & (1-p)^n\,\rho_{22} \end{pmatrix}$$

Since $(1-p)^n = \left(1 - \dfrac{\Gamma t}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-\Gamma t}$

$$\Rightarrow \mathcal{E}^{n}(\rho) = \begin{pmatrix} \rho_{11} + (1-e^{-\Gamma t})\rho_{22} & e^{-\frac{\Gamma t}{2}}\rho_{12} \\ e^{-\frac{\Gamma t}{2}}\rho_{21} & e^{-\Gamma t}\rho_{22} \end{pmatrix}$$

Off-diagonal elements decay slow (with e^{-t/T_2}) than diagonal element ρ_{22} (with e^{-t/T_1}), where $T_2 = 2T_1$.

(iii) Choi matrix $\rho_{\mathcal{E}}$ (in col-vec):

$$\rho_{\mathcal{E}} = (I \otimes \mathcal{E}) |\tilde{\Phi}\rangle\langle\tilde{\Phi}| =$$

$$= \sum_{i,j} |i\rangle_E\langle j|_E \otimes \mathcal{E}(|i\rangle_A\langle j|_A) =$$

$$= |0\rangle_E\langle 0|_E \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes \sqrt{1-p}\, |0\rangle\langle 1|$$

$$+ |1\rangle\langle 0| \otimes \sqrt{1-p}\, |1\rangle\langle 0|$$

$$+ |1\rangle\langle 1| \otimes \left[p\, |0\rangle\langle 0| + (1-p)\, |1\rangle\langle 1| \right]$$

Calculational details:

$$\mathcal{E}(|0\rangle\langle 0|_A) = |0\rangle\langle 0|$$

$$\mathcal{E}(|0\rangle\langle 1|) = \sqrt{1-p}\,|0\rangle\langle 1|$$

$$\mathcal{E}(|1\rangle\langle 0|) = \sqrt{1-p}\,|1\rangle\langle 0|$$

$$\mathcal{E}(|1\rangle\langle 1|) = p\,|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

$$\Rightarrow \rho_{\mathcal{E}} = \begin{array}{c} \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array} \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ \sqrt{1-p} & 0 & 0 & (1-p) \end{pmatrix} \frac{1}{2}$$

to have $\mathrm{Tr}\,\rho_{\mathcal{E}} = 1.$

(iv) Process χ matrix in Pauli basis

$\{I,X,Y,Z\}^{\otimes n} \ni P_b$

Expand Kraus operators in Pauli basis $M_a = \sum_b c_{ab} P_b$

$$\mathcal{E}(\rho) = \sum_{i,j} \chi_{ij} P_i \rho \overbrace{P_j^\dagger}^{=P_j} \qquad \rightarrow c_{ab} = \frac{1}{d} \mathrm{Tr}\left[M_a P_b\right]$$

Here:
$$M_0 = \frac{1}{2}\left(1+\sqrt{1-p}\right) I + \frac{1}{2}\left(1-\sqrt{1-p}\right) Z$$
$$M_1 = \sqrt{p}\, \sigma^- = \frac{\sqrt{p}}{2}(X - iY)$$

$$\Rightarrow \mathcal{E}(\rho) = \left(c_0 I + c_3 P_3\right) \rho \left(c_0 I + c_3 P_3\right) + \left(d_1 X + d_2 Y\right) \rho \left(d_1^* X + d_2^* Y\right)$$

$$\Rightarrow \chi_{ij} = \frac{1}{4} \begin{pmatrix} \left(1+\sqrt{1-p}\right)^2 & 0 & 0 & p \\ 0 & p & -ip & 0 \\ 0 & ip & p & 0 \\ p & 0 & 0 & \left(-1+\sqrt{1-p}\right)^2 \end{pmatrix}$$

Note the process χ matrix can have complex entries (in contrast, the PTM is purely real).

(v) Louiville superoperator (in col-vec basis)

$$\mathcal{L}_{\mathcal{E}} = \sum_a M_a^* \otimes M_a =$$

$$= \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} \otimes \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ & & & (1-p) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}$$

(vi) <u>Pauli transfer matrix</u> (Pauli-Liouville representation)

$$(R_\mathcal{E})_{ij} = \frac{1}{d}\,\mathrm{Tr}\left[P_i\,\mathcal{E}(P_j)\right]$$

$$\Rightarrow (R_\mathcal{E}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ p & 0 & 0 & 1-p \end{pmatrix}$$

use Mathematica (or Python)

Amplitude damping channel is <u>not</u> unital, i.e. it does not preserve the fully mixed state $\mathcal{E}(I) \neq I$.

Action on single qubit state:

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \qquad = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix}$$

$$\Rightarrow \mathcal{E}(\rho) = \sum (R_\mathcal{E})_{ij}\,\overbrace{|\rho\rangle\rangle_j} \equiv |\rho'\rangle\rangle_i$$

$$\Rightarrow \; |\rho')) = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{1-p}\; r_x \\ \sqrt{1-p}\; r_y \\ p + (1-p)\, r_z \end{pmatrix}$$

non-unital part ≙ shift of Bloch ellipsoid

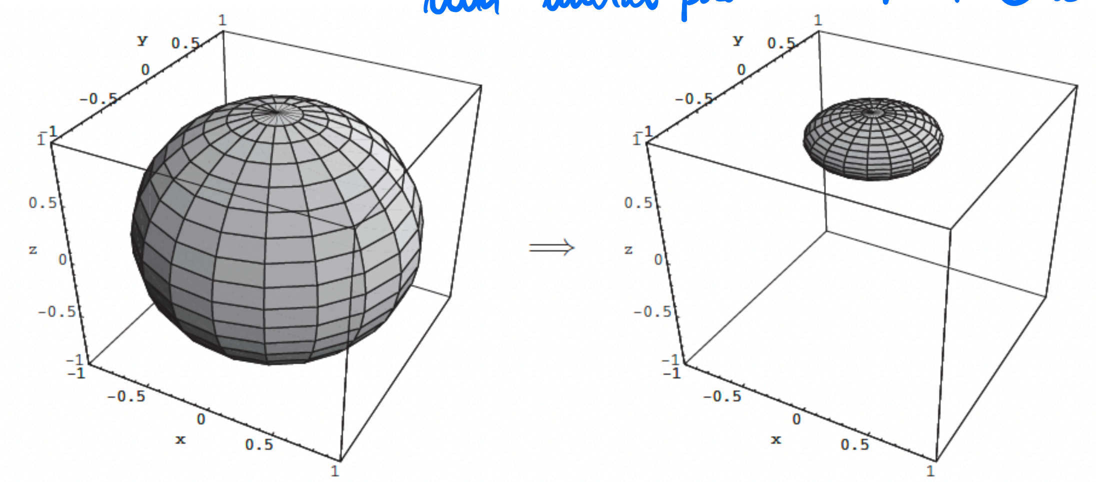

Figure 8.14. The effect of the amplitude damping channel on the Bloch sphere, for $p = 0.8$. Note how the entire sphere shrinks towards the north pole, the $|0\rangle$ state.

After many applications, we find

$$|\mathcal{E}^n(\rho)) = \frac{1}{2}\begin{pmatrix} 1 \\ e^{-\Gamma t/2}\; r_x \\ e^{-\Gamma t/2}\; r_y \\ 1 - e^{-\Gamma t}\; r_z \end{pmatrix} \xrightarrow[t\to\infty]{} \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Amplitude damping thus drives system to the north pole ($|0\rangle$) state.