Contents:

- Relation of Markovian master equation and quantum channels
- Quantum state tomography
- Quantum gate tomography
- Randomized benchmarking

Relation between Markovian master equations & CPTP maps

Closed system dynamics described by Schrödinger equation

$$\frac{d}{dt}\rho(t) = \frac{1}{i}[H, \rho(t)] = \mathcal{L}_H(\rho)$$

$\mathcal{L}_H$: Liouvillian

Open system dynamics (system coupled to environment) is generally non-local in time:

$$\frac{d}{dt}\rho(t) = \int d\tau\, \mathcal{L}[\rho(t)]$$

This is due to the memory of the bath. Memory is kept over timescale of bath correlation time τ_{corr}.

If τ_{corr} is much shorter than typical system evolution time t_{sys} (given by energy distances of occupied states):

$\tau_{corr} \ll t_{sys}$, then a time-local ($\equiv$ Markovian) description of open system dynamics is approximately valid:

$$\frac{d}{dt}\rho(t) = \mathcal{L}_\mathcal{E}[\rho(t)]$$

← Liouvillian of CPTP map $\mathcal{E}$

Evolution over infinitesimal time interval dt is described by CPTP map ($\hat{=}$ quantum channel) $\mathcal{E}$:

$$\rho(t+dt) = \mathcal{E}_{dt}[\rho(t)] = \sum_a M_a \rho(t) M_a^\dagger$$

$$= \rho(t) + dt\, \mathcal{L}_\mathcal{E}[\rho(t)]$$

Relation b/w Lindbladian $\mathcal{L}_\mathcal{E}$ and CPTP map $\mathcal{E}$ thus

$$\mathcal{E}_{dt} = \mathbb{I} + \mathcal{L}_\mathcal{E}\, dt$$

↑

after each infinitesimal time evolution of joint system+environment, the environmental state is discarded (Tr_E) and replaced by a fresh (equilibrium) state of the bath unentangled with the system.

Expanding $\mathcal{E}_{dt}$ in Kraus operators

$$\sum_a M_a \rho(t) M_a^\dagger = (\mathbb{I} + dt\, \mathcal{L})[\rho(t)]$$

Without loss of generality, we can choose

Hermitian operators

$$M_0 = I + O(dt) = I + (-iH + K)\,dt$$

$$M_a = \sqrt{dt}\, L_a \qquad \text{for } a \neq 0 .$$

Operators H, K, L_a are all zeroth order in dt.

We can determine K from the completeness relation

$$I = \sum_a M_a^\dagger M_a$$

$$= [I + (iH + K)\,dt][I + (-iH + K)\,dt]$$

$$+ dt \sum_a L_a^\dagger L_a =$$

$$= I + dt \underbrace{\left[2K + \sum_{a>0} L_a^\dagger L_a \right]}_{\stackrel{!}{=} 0} + O(dt)^2$$

$$\Rightarrow \quad \boxed{K = -\frac{1}{2} \sum_{a>0} L_a^\dagger L_a}$$

We thus obtain the Lindblad master equation (most general Markovian ME describing CPTP evolution):

$$\frac{d}{dt}\rho(t) = \frac{\rho(t+dt) - \rho(t)}{dt} = \frac{1}{dt}\left[\mathcal{E}_{dt}[\rho(t)] - \rho(t)\right] =$$

$$= \frac{1}{dt}\left[M_0 \rho M_0^\dagger + \sum_{a>0} M_a \rho M_a^\dagger - \rho(t)\right] =$$

$$= \frac{1}{dt}\left\{\left(I + (-iH + K)dt\right)\rho\left(I + (iH + K)dt\right)\right.$$

$$\left. + dt \sum_{a>0} L_a \rho L_a^\dagger - \rho(t)\right\} =$$

$$\left(K = -\tfrac{1}{2}\sum_{a>0} L_a^\dagger L_a\right)$$

$$\Leftrightarrow \boxed{\dot{\rho}(t) = -i[H, \rho] + \sum_{a>0}\left(L_a \rho L_a^\dagger - \frac{1}{2} L_a^\dagger L_a \rho - \frac{1}{2}\rho L_a^\dagger L_a\right)}$$

Lindblad master equation

Similarly to the Kraus operators there exist a unitary freedom in choice of Lindblad operators

$$\tilde{L}_a = \sum_b V_{ba} L_a \qquad (V \text{ unitary})$$

Different choices of Lindblad "jump" operators correspond to different unravelings of the same Markovian dynamics.

Probability for jump event a to occur during dt:

$$\text{Prob}(a) = dt \langle \psi(t) | L_a^\dagger L_a | \psi(t) \rangle$$

$\Rightarrow$ if jump occurs the updated wavefunction is

$$|\psi(t+dt)\rangle = \frac{L_a |\psi(t)\rangle}{\| L_a |\psi(t)\rangle \|}$$

if no jump occurs the state evolves as

$$|\psi(t+dt)\rangle = \frac{M_0 |\psi(t)\rangle}{\| M_0 |\psi(t)\rangle \|} .$$

Stochastic Schrödinger equation evolution $\rightarrow$ average over many different paths (sampling possible jump event paths).

Quantum state tomography (QST)

Procedure of experimentally determining an unknown quantum state.

Task: Measuring the density operator ρ given many copies of the state.

Reconstruct ρ from measuring its components in a basis of measurement operators E_j, $j = 1, \dots, d^2$ ($d = 2^n$) that span the n-qubit Hilbert space $\mathcal{L}(\mathcal{H}_d)$.

Typical choice is the Pauli basis $|E_j\rangle\!\rangle = |j\rangle\!\rangle_\sigma$, where $|j\rangle\!\rangle_\sigma \hat{=}$ vectorized element $P_j \in \{I, X, Y, Z\}^{\otimes n}$.

Generally, one measures the probabilities

$$p_j = \langle\!\langle E_j | \rho \rangle\!\rangle = \sum_k \underbrace{\langle\!\langle E_j | k \rangle\!\rangle}_{\text{matrix } A_{j,k} \text{ is known}} \underbrace{\langle\!\langle k | \rho \rangle\!\rangle}_{\text{Coefficient of } \rho \text{ in Pauli basis } P_k}$$

(choice of experiments) and must be invertible.

Can estimate p_j as sample average of measurement outcomes:

$$m_j = \frac{1}{N} \sum_{i=1}^{N} m_{ij}$$

$m_{ij} \in \{0, 1\}$ is outcome of i-th measurement of E_j.

m_{ij} ↑ average over shots (measurements of N copies of ρ)

Expected value of m_j is

$$\mathbb{E}(m_j) = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\mathbb{E}(m_{ij})}_{= p_0 \cdot 0 + p_1 \cdot 1 = p_1} = p_j$$

$\mathbb{E}(X) := \sum_j p_j x_j$ — p_j: probabilities; x_j: random variables

Variance:

$$\mathrm{Var}(m_j) = \mathbb{E}\left[(m_j - \mathbb{E}(m_j))^2 \right] =$$

$$= \mathbb{E}(m_j^2) - 2\,\mathbb{E}(m_j)^2 + \mathbb{E}(m_j)^2 =$$

$$= \underbrace{\mathbb{E}(m_j)}_{= p_j} - 2p_j^2 + p_j^2 = p_j - p_j^2 =$$

$$= p_j(1 - p_j)$$

$\mathbb{E}(m_j^2) = p_0 \cdot 0^2 + p_1 1^2 = p_1 = \mathbb{E}(m_j)$ ← $\mathbb{E}(m_j)$

The variance (or error) of the mean:

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{N}\sum_i X_i\right) = \frac{1}{N^2}\mathrm{Var}\left(\sum_i X_i\right) =$$

general property of the variance

$$= \frac{1}{N^2}\sum_i \mathrm{Var}(X_i) =$$

uncorrelated variables, like uncorrelated (IID) measurements

$$\Rightarrow \mathrm{Var}(\bar{X}) = \frac{\mathrm{Var}(X)}{N}$$ (if all $\mathrm{Var}(X_i)$ are the same).

Here:

$$\mathrm{Var}(\bar{m}_j) = \frac{p_j(1-p_j)}{N}$$

Numerator $p_j(1-p_j)$ is also called single-shot variance.

The sample average m_j thus approaches the true probability value p_j as the number of measurements $N \to \infty$.

Then, $\hat{s}$ is linear inversion estimator of s; vector of measurements m_j

$$|\hat{s}\rangle\rangle = A^{-1}\,|m\rangle\rangle \quad , \quad A_{jk} = \langle\langle E_j | P_k \rangle\rangle$$

$$\Leftrightarrow \hat{s}_j = \sum_k \left(A^{-1}\right)_{jk} m_k$$

Note that we can extract expectation values of $P_0 \equiv I$ and $P_j = Z$ in same basis measurement $\Rightarrow$ $3^n - 1$ measurements are necessary to estimate m_j, $P_j \in \{I, X, Y, Z\}^{\otimes n}$.

One coefficient is fixed by normalization of the state.

If A is not square since we are using an over-complete measurement basis $|E_j\rangle\rangle$ with $j > d^2$, then we obtain $\hat{\rho}$ as pseudo-inverse (minimizes least-squares)

$$|\hat{\rho}\rangle\rangle = (A^T A)^{-1} A^T |m\rangle\rangle$$

Quantum process tomography:

Goal: Characterize gate G (instead of state ρ).

Done by measuring process matrix χ (e.g. in Pauli or one-hot $|i\rangle\langle j|$ basis) or PTM $(R_\mathcal{E})_{kl} = \langle\langle k|G|l\rangle\rangle_\rho$.

This is achieved by measuring the d^4 probabilities

gate that we want to estimate χ or PTM $R_\mathcal{E}$ for.

$$p_{ji} = \langle\langle E_j | G | \rho_i \rangle\rangle$$

Basis of measurement operators (fiducial basis, informationally-complete)

$\Rightarrow$ the probabilities $p_i = \mathrm{Tr}(\rho E_i)$ completely characterize the state.

One can distinguish measurement outcomes j from POVMs m and label $E_j^{(m)}$. E.g. $m = \{X, Y, Z\}$ basis measurements with outcomes $i = 0, 1$. Then E_i contains all of the $E_j^{(m)}$.

Set of vectors $|\rho_i\rangle\rangle$ forms a complete basis for the Hilbert-Schmidt space. Note that we cannot choose $|\rho_i\rangle\rangle$ as $|i\rangle\rangle_\rho$ as $\mathrm{Tr}[|\rho_i\rangle\rangle] = 1$, but $\mathrm{Tr}(|i\rangle\rangle_\rho) = 0$.

Instead, each $|\rho_i\rangle\rangle$ must be chosen as some linear combination of Pauli basis vectors $|\rho_i\rangle\rangle = \sum_j c_{ij} |j\rangle\rangle_\rho$.

Inserting two complete sets of Pauli basis states

$$p_{ji} = \langle\langle E_j | G | \rho_i \rangle\rangle$$

$$\Leftrightarrow \quad p_{ji} = \sum_{k,l} \underbrace{\langle\langle E_j | k \rangle\rangle}_{\text{known projection of measurement effects } \langle\langle E_j| \text{ on Pauli basis}} \overbrace{\langle\langle k | G | l \rangle\rangle}^{=\text{PTM}} \underbrace{\langle\langle l | \rho_i \rangle\rangle}_{\text{known coefficients of } |\rho_i\rangle\rangle \text{ in Pauli basis}}$$

Now vectorize the PTM to write this as a matrix equation:

$$g_{k + (l-1)d^2} = \langle\langle k | G | l \rangle\rangle \quad \text{(column stacking)}$$

$$S_{j+(i-1)d^2,\, k+(l-1)d^2} = \langle\langle E_j | k \rangle\rangle \langle\langle l | \rho_i \rangle\rangle$$

$$p_{j+(i-1)d^2} = p_{ji}$$

$$\Rightarrow \quad \vec{p} = S\vec{g}$$

If S is a square matrix (it has full rank by definition), then

$$\vec{g} = S^{-1}\vec{p}$$

Here, $\vec{p}$ = vector of $\hat{p}_i = \frac{1}{N}\sum_s m_{si}$ (estimates of p_i).

If the sets of measurement operators $\{E_j\}$ or input states $\{\rho_i\}$ are overcomplete, the S matrix has more rows than columns, $i,j > d^2$. Then we obtain $\vec{g}$ by the pseudo-inverse of S (this corresponds to the least-squares solution of $S\vec{g} = \vec{p}$):

$$\boxed{\vec{g} = (S^T S)^{-1} S^T \vec{p}}$$

Note, without vectorization, we can write

$$p_{ji} = \sum_{k,l} A_{jk}\, G_{kl}\, B_{li}$$

$A_{jk} = \langle\langle E_j | k \rangle\rangle$ $\qquad$ $B_{li} = \langle\langle l | \rho_i \rangle\rangle$

If the inverses of A and B exist, then

$$G = A^{-1} P B^{-1}$$

If not, we can use the pseudo-inverses and write

$$G = (A^T A)^{-1} A^T P B^T (B B^T)^{-1}$$

Example: Quantum process tomography of a single qubit

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$$

$$\mathcal{E}(\rho) = \sum_a M_a \rho M_a^\dagger \quad , \quad \sum_a M_a^\dagger M_a = I$$

We can write $M_a = \sum_k c_{ak} P_k = \sum \tilde{c}_{ak} E_k$, with $P_k = |k\rangle\rangle_P$ and $E_k = |k\rangle\rangle_C$

$$\Rightarrow M_a^\dagger = \sum_k c_{ak}^* P_k = \sum_k \tilde{c}_{ak}^* E_k \Big| |k\rangle\rangle_P$$

$E_k = E_{i+(j-1)d} \equiv$
$= |i\rangle\langle j|$
$= |i,j\rangle\rangle$

Thus,

$$\mathcal{E}(\rho) = \sum_a \sum_{k,l} c_{ak} c_{al}^* P_k \rho P_l$$

$$= \sum_{k,l} \chi_{kl} P_k \rho P_l^\dagger \equiv \rho'$$

↖ process matrix in Pauli basis's

with process matrix $\chi_{kl} = \sum_a c_{ak} c_{al}^*$.

Can use any basis, $P_k = |k\rangle\rangle_P$, $E_k = |k\rangle\rangle_C$, or any other.

Now expand $\rho' = \mathcal{E}(\rho)$ in any basis & do state tomography.

Choose $|\rho_i\rangle\rangle$ as follows:

$$\rho_1 = |0\rangle\langle 0|$$

$$\rho_2 = |1\rangle\langle 1|$$

$$\rho_3 = |+\rangle\langle +| \quad \text{with} \quad |+\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$$

$$\rho_4 = |-\rangle\langle -| \quad \text{with} \quad |-\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle]$$

Then,

$$\rho_3 \pm i\rho_4 - \frac{1 \pm i}{2}\rho_1 - \frac{1 \pm i}{2}\rho_2 =$$

$$= |+\rangle\langle +| \pm i|-\rangle\langle -| - \frac{1 \pm i}{2}|0\rangle\langle 0| - \frac{1 \pm i}{2}|1\rangle\langle 1| =$$

$$= \frac{1}{2}[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|]$$

$$\pm \frac{i}{2}[|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|]$$

$$- \frac{1 \pm i}{2}[|0\rangle\langle 0| + |1\rangle\langle 1|] =$$

$= |0\rangle\langle 1| \; (\; |1\rangle\langle 0| \;).$

By preparing the states $|0\rangle, |1\rangle, |+\rangle, |-\rangle$, we can thus determine

$$\mathcal{E}(|0\rangle\langle 1|) = \mathcal{E}(|+\rangle\langle +|) + i\,\mathcal{E}(|-\rangle\langle -|) - \frac{1+i}{2}\left[\mathcal{E}(|0\rangle\langle 0|) + \mathcal{E}(|1\rangle\langle 1|)\right].$$

Of course, $\mathcal{E}(|1\rangle\langle 0|) = \mathcal{E}^*(|0\rangle\langle 1|)$ and the diagonal elements are obtained from preparing $|0\rangle$ and $|1\rangle$.

We thus have used the input states ρ_i to determine

and can express $|i\rangle\!\rangle_C = \sum_j C_{ij}\, \rho_j$

column stacked basis $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

with coefficients $C\langle\!\langle j|\rho_i\rangle\!\rangle = C_{ij}$

$$C_{1j} = (1, 0, 0, 0)$$

$$C_{2j} = \left(-\frac{1-i}{2}, -\frac{1-i}{2}, 1, -i\right)$$

$$C_{3j} = \left(-\frac{1+i}{2}, -\frac{1+i}{2}, 1, i\right)$$

$$C_{4j} = (0, 1, 0, 0).$$

We can also easily express the Pauli basis in terms of the ρ_i (i.e., $\rho\langle\langle j | \rho_i \rangle\rangle$ or $|j\rangle\rangle_\rho = \sum_i c_{ji}\, \rho_i$.

$$P_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_1 + \rho_2$$

$$P_1 = X = \left(\rho_3 + i\rho_4 - \frac{1+i}{2}\rho_1 - \frac{1+i}{2}\rho_2\right) + \text{c.c.} =$$

$$= 2\rho_3 - \rho_1 - \rho_2 =$$

$$P_2 = Y = -i\left(\rho_3 + i\rho_4 - \frac{1+i}{2}\rho_1 - \frac{1+i}{2}\rho_2\right) + \text{c.c.}$$

$$= 2\rho_4 - \rho_1 - \rho_2$$

$$P_3 = Z = \rho_1 - \rho_2$$

Then measure these states $\mathcal{E}(\rho_i)$ in the Pauli basis P_j to experimentally obtain p_{ji}:

$$p_{ji} = \mathrm{Tr}\left[P_j\, \mathcal{E}(\rho_i)\right] = A\, G\, B$$

Alternatively:

$$|\rho_i\rangle\!\rangle = \sum_k B_{ki}\,|k\rangle\!\rangle$$

$$P_{ji} = \sum_{k,l} A_{jk}\,\sigma_{kl}\,B_{li}$$

$A_{jk} = \langle\!\langle E_j | k \rangle\!\rangle$; $B_{li} = \langle\!\langle l | \rho_i \rangle\!\rangle$ → from above, we can read off:

$\rho_1 = \frac{1}{2}(I+Z)$; $\rho_2 = \frac{1}{2}(I-Z)$

$\rho_3 = \frac{1}{2}(I+X)$; $\rho_4 = \frac{1}{2}(I+Y)$

$$\Rightarrow A_{jk} = \delta_{jk}$$

$$B_{li} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

Since

$$P_k \in \left\{ \frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}} \right\}^{\otimes n}$$

$$\Rightarrow \rho_1 = \frac{1}{\sqrt{2}}(P_0+P_3)$$

$$\rho_2 = \frac{1}{\sqrt{2}}(P_0-P_3)$$

$$\rho_3 = \frac{1}{\sqrt{2}}(P_0+P_1)$$

$$\rho_4 = \frac{1}{\sqrt{2}}(P_0+P_2)$$

$$\Rightarrow B^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\Rightarrow \sigma = A^{-1} P B^{-1} = P B^{-1}.$$

$$A^{-1} = I$$

where $P = P_{ji} = \mathrm{Tr}\left[P_j\, \mathcal{E}(\rho_i)\right]$.

Example: Depolarizing noise channel with $p \equiv \lambda = 0.1$.

$$\mathcal{E}(\rho_1) = \mathcal{E}(|0\rangle\langle 0|) = \frac{1}{2}(I + \vec{r}' \cdot \vec{\sigma})$$

$$\text{with } \vec{r}' = \begin{pmatrix} 0 \\ 0 \\ 1 - \frac{4}{3}\lambda \end{pmatrix} = \frac{1}{2}\left(I + \left(1 - \frac{4}{3}\lambda\right) Z\right)$$

$$\Rightarrow P_{j1} = \left(1, 0, 0, 1 - \frac{4}{3}\lambda\right)^T \quad (j = 0, \ldots, 3)$$

$$\mathcal{E}(\rho_2) = \mathcal{E}(|1\rangle\langle 1|) = \frac{1}{2}(I + \vec{r}' \cdot \vec{\sigma})$$

$$\text{with } \vec{r}' = \begin{pmatrix} 0 \\ 0 \\ -(1 - \frac{4}{3}\lambda) \end{pmatrix}$$

$$\Rightarrow P_{j2} = \left(1, 0, 0, -\left(1 - \frac{4}{3}\lambda\right)\right)^T$$

$$\mathcal{E}(\rho_3) = \mathcal{E}(|+\rangle\langle +|) = \frac{1}{2}(I + \vec{r}' \cdot \vec{r}) \quad \vec{r} = (1,0,0)$$

$$\text{with } \vec{r}' = \begin{pmatrix} 1 - \frac{4}{3}\lambda \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow P_{j3} = \left(1, 1 - \frac{4}{3}\lambda, 0, 0\right)^T$$

$$\mathcal{E}(\rho_4) = \mathcal{E}(|1\rangle\langle -1|) \overset{\checkmark}{\cong} \frac{1}{2}(I + \vec{r}' \cdot \vec{\sigma}) \qquad \vec{r} = (0,1,0)$$

$$\text{with } \vec{r}' = \begin{pmatrix} 0 \\ 1-\frac{4}{3}\lambda \\ 0 \end{pmatrix}$$

$$\Rightarrow P_{j4} = \left(1, 0, 1-\tfrac{4}{3}\lambda, 0\right)^T$$

$$\Rightarrow P = (P_{ji}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1-\frac{4}{3}\lambda & 0 \\ 0 & 0 & 0 & 1-\frac{4}{3}\lambda \\ 1-\frac{4}{3}\lambda & -1+\frac{4}{3}\lambda & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

due to normalization

$P_{jk} \in \left\{\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}}\right\}$

$$\Rightarrow G = P B^{-1} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{4}{3}\lambda & 0 & 0 \\ 0 & 0 & 1-\frac{4}{3}\lambda & 0 \\ 0 & 0 & 0 & 1-\frac{4}{3}\lambda \end{pmatrix}$$

as expected.

Randomized benchmarking

Model-free benchmarking technique that yields the average fidelity F_{av} over the full set of Clifford gates Clif_n. The Clifford group is the normalizer of the Pauli group $\mathcal{P}$, i.e., it maps Pauli operators $P_m \in \mathcal{P}$ to Pauli operators: $g P_m g^{-1} = P_m \quad \forall g \in \mathrm{Clif}_n$.

The Clifford group is generated by $\{H, S, \mathrm{CNOT}\}$ and circuits composed only out of Clifford gates can be efficiently simulated classically (Gottesman-Knill theorem).
Clifford group plays important role in QEC using stabilizer codes and forms a unitary 2-design, which implies

$$\frac{1}{|\mathrm{Clif}_n|} \sum_{j=1}^{|\mathrm{Clif}_n|} \left(C_j \Lambda (C_j^\dagger \rho C_j) C_j^\dagger \right) =$$

w/o proof; we show a similar statement explicitly later for Pauli twirls

uniform Haar measure of $U(d)$

$$= \int_{U(d)} dU \left(U \Lambda (U^\dagger \rho U) U^\dagger \right) = \Lambda_{dep}(\rho)$$

Λ_{dep} is depolarizing channel with same average fidelity F_{ave} as Λ. Now, $\Lambda_{dep} = (1-\tilde{p})\rho + \tilde{p}\frac{I}{d}$ and thus $F_{ave} = 1 - \tilde{p} + \frac{\tilde{p}}{d}$ (shown below).

RB is scalable in the number of qubits n ($O(n^2)$ gates and $O(n^4)$ classical preprocessing costs). However, RB only provides partial information about the noise (= error channel), specifically the average fidelity F_{ave}. This is related to the χ_{00} element of the χ process matrix as

$$F_{ave} = \frac{\chi_{00}\, d + 1}{d+1}.$$

RB is unaffected by SPAM errors (unlike quantum process tomography), while SPAM refers to state preparation & measurement errors. RB in fact gives estimates for SPAM errors.

Fidelity of depolarizing channel:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) =$$

$$= \left(1 - \frac{3}{4}\tilde{p}\right)\rho + \frac{\tilde{p}}{4}(X\rho X + Y\rho Y + Z\rho Z) =$$

$$= (1-\tilde{p})\rho + \tilde{p}\frac{I}{d} \qquad (d = 2 \text{ here, generally } 2^n).$$

Here, $\frac{3}{4}\tilde{p} = p \Leftrightarrow \tilde{p} = \frac{4}{3}p$.

The average fidelity of the depolarizing channel is defined as

$$\overline{F}_{ave}(\mathcal{E}) = \int d\psi \; F\left(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)\right)$$

↑ Haar measure

with state-dependent channel

$$F\left(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)\right) = \langle\psi|\,\mathcal{E}(|\psi\rangle\langle\psi|)\,|\psi\rangle.$$

Inserting the depolarizing channel yields (use def. w/o square root here)

$$\langle\psi|\left[(1-\tilde{p})\underbrace{\rho}_{=|\psi\rangle\langle\psi|} + \tilde{p}\frac{I}{d}\right]|\psi\rangle = 1 - \tilde{p} + \underbrace{\frac{\tilde{p}}{d}}_{=\tilde{p}/2} = F\left(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)\right)$$

RB protocol:

Generate random sequences of Clifford gates $C_i \in \text{Clif}_m$ of length $m \leq M-1$. Fix initial state $|\psi\rangle$ and maximal sequence length M.

Step 1: fix $m \leq M-1$ and generate K_m sequences consisting of $(m+1)$ quantum operations.

The first m are chosen uniformly random from Clif_m.

The $(m+1)$th operation is chosen such that (possible as Clif_m is a group)

$$C_{i_{m+1}} \circ \left[C_{i_m} \circ \cdots C_{i_1} \right] = \mathbb{I}$$

Each Clifford gate is associated with an error channel $\Lambda_{i_j,j}$ such that the sequence K_m corresponds to the quantum operation

$$S_{\vec{i}_m} = \bigcirc_{j=1}^{m+1} \left(\Lambda_{i_j,j} \circ C_{i_j} \right) \,, \quad \vec{i}_m = (i_1, \ldots, i_m)$$

Note that i_{m+1} is uniquely determined by $\vec{i}_m$.

Step 2: For each of K_m sequences, measure the survival probability

$$\mathrm{Tr}\left[E_\psi \, S_{\vec{i}_m}(\rho_\psi)\right]$$

ρ_ψ = noisy version of $|\psi\rangle\langle\psi|$ (state prep. error) } SPAM
E_ψ = noisy measurement of $|\psi\rangle\langle\psi|$ (measurement error) } error

Step 3: Average over K_m random realizations to find the averaged sequence fidelity

$$F_{\mathrm{seq}}(m,\psi) = \mathrm{Tr}\left[E_\psi \, S_{K_m}(\rho_\psi)\right]$$

where

$$S_{K_m} = \frac{1}{K_m} \sum_{\vec{i}_m} S_{\vec{i}_m}$$

Step 4: Repeat steps 1 through 3 for different values of m and fit results to the model

$$F_g(m, \psi) = A_0 (1-\rho)^m + B_0$$

A_0: state preparation error
B_0: measurement error
} SPAM error

The average error rate $r = 1 - F_{av} =$

$$= 1 - \left[1 - \tilde{\rho} + \frac{\tilde{\rho}}{d}\right] =$$

$$= \tilde{\rho} - \frac{\tilde{\rho}}{d}. \quad (d = 2^n \text{ for } n \text{ qubits})$$

Note that it becomes obvious that we measure the twirled channel over the Clifford group, i.e., a depolarizing channel with the same average fidelity as the original channel.

For this define: (1) $\mathcal{D}_{i_1} = C_{i_1}$

(2) $\mathcal{D}_{i_2} = C_{i_2} \circ C_{i_1} \Rightarrow C_{i_2} = \mathcal{D}_{i_2} \circ \mathcal{D}_{i_1}^{\dagger}$

(3) given if $C_{i_n}, \dots, C_{i_j}$ have been chosen and $\mathcal{D}_{i_n}, \dots, \mathcal{D}_{i_j}$ have been defined accordingly, define $\mathcal{D}_{i_{j+1}}$ such that $C_{i_{j+1}} = \mathcal{D}_{i_{j+1}} \circ \mathcal{D}_{i_j}$ (i.e.,

$$\mathcal{D}_{i_{j+1}} = C_{i_j} \circ \cdots \circ C_{i_n} .$$

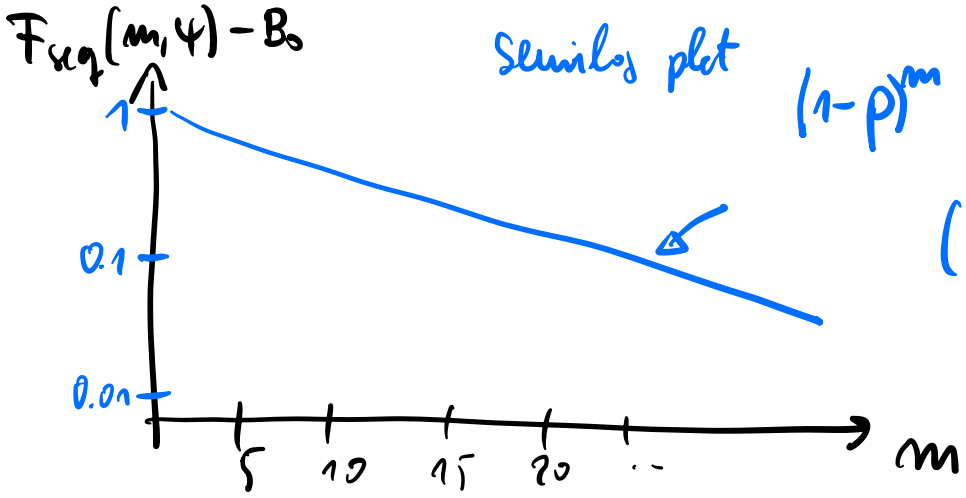
Then,

$$S_{\vec{i}_m} = \Lambda_{i_{m+1}} \circ C_{i_{m+1}} \circ \Lambda_{i_m} \circ C_{i_m} \circ \cdots \Lambda_{i_1} \circ C_{i_1} =$$

$$= \Lambda_{i_{m+1}} \circ \mathcal{D}_{i_{m+1}} \circ \mathcal{D}^{\dagger}_{i_m} \circ \Lambda_{i_m} \circ \mathcal{D}_{i_m} \circ \mathcal{D}^{\dagger}_{i_{m-1}} \circ \Lambda_{i_{m-1}} \circ \mathcal{D}_{i_{m-1}}$$

$$\circ \cdots \circ \underbrace{\mathcal{D}^{\dagger}_{i_1} \circ \Lambda_{i_1} \circ \mathcal{D}_{i_1}}_{\text{Clifford twirled channel}} .$$

$\Rightarrow$ depolarizing noise channel with same average fidelity p as original channel Λ_{i_j}.



$$(1-p)^m = \exp[m \log(1-p)]$$

$$(1-p) \approx 1 - \varepsilon.$$

Forest benchmarking suite

Idea of interleaved RB: determine av. error of particular Clifford gate G_i

(1) run original RB: K_m sequences $C_{i_{m+1}} \circ C_{i_m} \circ \cdots \circ C_{i_1}$

(2) for every sequence $\vec{i}_m$, run the circuit

$$\tilde{C}_{i_{m+1}} \circ G_i \circ C_{i_m} \circ \cdots \circ G_i \circ C_{i_2} \circ G_i \circ C_{i_1}$$

↑ needs to be updated compared to standard RB (1) to invert the sequence containing G_i.

⇒ compute r for (1) & (2) ⇒ difference is r_{G_i}.