Quantum Algorithms and Error Correction

Peter Orth

May 30, 2023

Contents

1 Basics: ρ, channels, noise

1.1 Density operator ρ (Review)

A general state of a quantum system is described by the *density operator*

$$\rho := \sum_i \underset{\text{prob. to find system in pure state } |\psi_i\rangle}{\underset{\uparrow}{p_i}} |\psi_i\rangle\langle\psi_i|. \tag{1}$$

The $|\psi_i\rangle$ are normalized, so $\langle\psi_i|\psi_i\rangle = 1$, but do not need to be orthogonal.

We write an ensemble of pure states as $\mathcal{E} = \{p_i| |\psi_i\rangle\}$

Properties of the density operator

The density operator has to fulfill two properties to describe an actual ensemble.

(i) $\operatorname{Tr}\rho = 1$.

(ii) ρ is a positive operator, so $\langle\psi|\rho|\psi\rangle \geq 0 \ \forall|\psi\rangle$.

Proof:

$\rightarrow$: Suppose $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is a density operator. Then,

(i) We get

$$\operatorname{Tr}\rho = \sum_i \underbrace{\operatorname{Tr}(|\psi_i\rangle\langle\psi_i|)}_{\sum_m \langle m|\psi_i\rangle\langle\psi_i|m\rangle = \langle\psi_i|\psi_i\rangle = 1} = \sum_i p_i = 1 \tag{2}$$

(ii) Take a general vector $|\varphi\rangle$, then

$$\langle\varphi|\rho|\varphi\rangle = \sum_i \overbrace{\langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle}^{=|\langle\psi|\varphi\rangle|} \geq 0 \tag{3}$$

$\leftarrow$: Suppose ρ is a positive operator with $\operatorname{Tr}\rho = 1$. Since it is a positive operator, it is also hermitian ($\rho^\dagger = \rho$) and therefore normal ($\rho^\dagger\rho = \rho\rho^\dagger$), which allows for spectral decomposition. This means that

$$\rho = \sum_i \underset{\geq 0}{\lambda_i} \underset{\text{orthonormal basis } \langle i|j\rangle = \delta_{ij}}{\underset{\uparrow}{|i\rangle}} \langle i| \tag{4}$$

Due to $\operatorname{Tr}\rho = 1 \Rightarrow \sum_i \lambda_i = 1 \Rightarrow \lambda_i$ can be interpreted as probabilities.
$\Rightarrow \rho$ describes an ensemble of states $\mathcal{E} = \{\lambda_i \equiv p_i||i\rangle\}$

■

Purity

A density operator ρ obeys $\operatorname{Tr}\rho^2 = 1$ iff ρ describes a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$
Proof:

$\Rightarrow$: Suppose $\rho = |\psi\rangle\langle\psi| \Rightarrow \rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = \rho$ and therefore $\operatorname{Tr}\rho^2 = \operatorname{Tr}\rho = 1$.

$\Leftarrow$: Take a density operator $\rho = \sum_i p_i|i\rangle\langle i|$ We can write

$$\operatorname{Tr}\rho^2 = \sum_{i,j,k} p_i p_j \underbrace{\langle k|i\rangle}_{\delta_{ki}}\langle i|i\rangle \underbrace{\langle j|k\rangle}_{\delta_{ik}} = \sum_i p_i^2 \tag{5}$$

We distinguish the cases

(1) only one $p_i \neq 0 \Rightarrow p_1 = 1, p_2 = \cdots = 0 \Rightarrow \operatorname{Tr}\rho^2 = 1$

(2) at least two $p_i \neq 0$: from $\sum_i p_i = 1$ we get that

$$\left(\sum_i p_i\right)^2 = 1 \Leftrightarrow \underline{\sum_i p_i^2} + \sum_{i\neq j} p_i p_j = 1 \tag{6}$$

$\Rightarrow \sum_i p_i^2 < 1$ if at least two $p_i > 0$ (mixed state).

Theorem: Density operators form a convex set

Given two density operators ρ_1 and ρ_2, the convex linear combination

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2 \text{ with } 0 \leq \lambda \leq 1 \tag{7}$$

is also a density operator

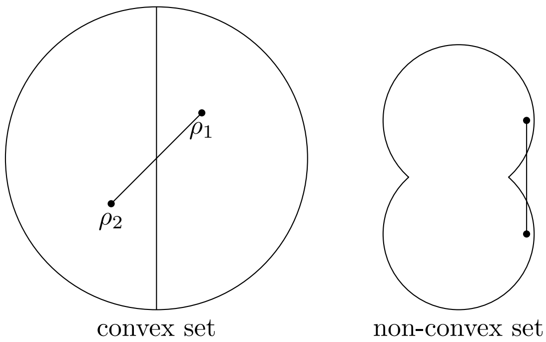

Figure 1: Examples for convex and non-convex sets: convexity means that for any two points, the connecting line is always in the set.

Proof:

(i) $\operatorname{Tr}\rho = \lambda \underbrace{\operatorname{Tr}\rho_1}_{=1} + (1-\lambda)\underbrace{\operatorname{Tr}\rho_2}_{=1} = 1$

(ii) $\langle\varphi|\rho|\varphi\rangle = \lambda \underbrace{\langle\varphi|\rho_1|\varphi\rangle}_{\geq 0} + (1-\lambda)\underbrace{\langle\varphi|\rho_2|\varphi\rangle}_{\geq 0} \geq 0$

Such a state $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ is called a mixture of states ρ_i.

Extremal points

States in a convex set can be expressed as a convex linear combination of its extremal states.
Here, the extremal states are pure states $|\psi_i\rangle\langle\psi_i|$, since $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Pure states are extremal points as they cannot be expressed as a sum of other states.
Proof:
Consider $\rho = |\psi\rangle\langle\psi|$ and let $|\psi_\perp\rangle$ be a vector perpendicular to $|\psi\rangle$: $\langle\psi_\perp|\psi\rangle = 0$, so

$$\langle\psi_\perp|\rho|\psi_\perp\rangle = 0 = \lambda \underbrace{\langle\psi_\perp\rho_1|\psi_\perp\rangle|}_{\geq 0} + (1-\lambda)\underbrace{\langle\psi_\perp\rho_2|\psi_\perp\rangle|}_{\geq 0} \tag{8}$$

$\Rightarrow$ either $\lambda = 0, 1$: $\rho_1 = \rho, \rho_2 = \rho$ or $\langle\psi_\perp|\rho_1|\psi_\perp\rangle = \langle\psi_\perp|\rho_2|\psi_\perp\rangle = 0 \;\forall\; |\psi\rangle$.

States on the boundary of the convex set of ρ

States ρ at the boundary have at least one zero eigenvalue, since there are states nearby with negative eigenvalues.

Example: single qubits

We have a general single-qubit pure state

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \text{ with } |\vec{r}| = 1, \vec{\sigma} = (X, Y, Z) \tag{9}$$

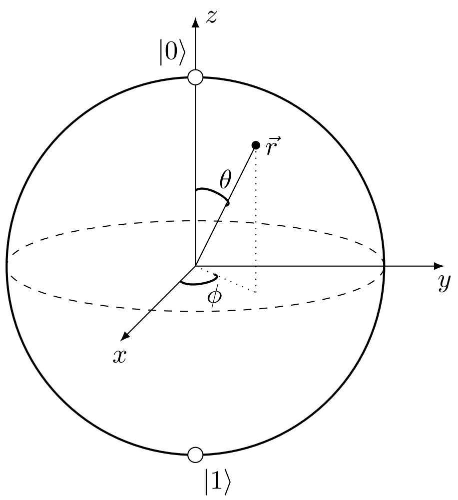

Figure 2: Bloch-sphere

- pure states have $|\vec{r}| = 1$,
- mixed states have $|\vec{r}| < 1$.

Postulate 1

Associated to any isolated quantum system is a Hilbert space $\mathcal{H}$ (complex vector space with an inner product). The system is completely characterized by the density operator ρ ($\operatorname{Tr}\rho = 1, \rho \geq 0$).
If the state is ρ_i with probability p_i, then

$$\rho = \sum_i p_i \rho_i, \; \sum_i p_i = 1 \tag{10}$$

Postulate 2

The time-evolution of a closed quantum system is described by a unitary transformation U with

$$\rho(t_2) = U(t_1, t_2)\rho(t_1)U^\dagger(t_1, t_2) \tag{11}$$

Explicitely we have

$$U(t_1, t_2) = T \exp\left[-i\int_{t_1}^{t_2} \mathrm{d}s \underbrace{H(s)}_{\substack{\uparrow \\ \text{Hamiltonian}}}\right] \tag{12}$$

Postulate 3

Measurements are described by a collection of measurement operators $\underset{\substack{\uparrow \\ m \text{refers to the possible measurement outcomes}}}{\{M_m\}}$ acting on the state space of the system.
If state before measurement is ρ, the result m is observed with probability $p(m) = \operatorname{Tr}\left(M_m^\dagger M_m \rho\right)$. The state after the measurement is

$$\rho \mapsto \frac{M_m \rho M_m^\dagger}{\operatorname{Tr}\left(M_m^\dagger M_m \rho\right)}, \; \sum_m M_m^\dagger M_m = \mathbb{1} \tag{13}$$

Special case: Projective measurement

Let $M_m = P_m$ with $P_m^2 = P_m = |m\rangle\langle m|$. We get $P_m^\dagger = P_m,\ P_m P_{m'} = P_m \delta_{mm'}$. This gives

$$\sum_m M_m^\dagger M_m = \sum_m P_m^\dagger P_m = \sum_m P_m = \mathbb{1} \tag{14}$$

We call an observable $M = \sum_m |m\rangle\langle m| = \sum_m m P_m$. We measure m with probability $p(m) = \mathrm{Tr}(\rho P_m)$ and the state afterwards becomes $\rho \mapsto \frac{P_m \rho P_m}{\mathrm{Tr}(P_m \rho)}$

POVM: positive operator-valued measurements

We define a POVM by $\{E_m\}$ where $E_m = M_m^\dagger M_m$, which are positive operators ($E_m \geq 0$).

They obex the completeness relation $\sum_m E_m = \mathbb{1}$. We get $p(m) = \mathrm{Tr}(E_m \rho)$.

Since $E_m \geq 0 \Rightarrow E_m^\dagger = E_m \Rightarrow$ allows spectral decomposition. However, $E_m E_{m'} \underset{\substack{\uparrow \\ \text{in general}}}{\neq} E_m \delta_{mm'}$.

2 Tomography: State & gate tomography

3 Quantum algorithm: Trotter

4 Variational quantum algorithms

5 Quantum error mitigation

6 Quantum error correction

7 Literature

- Preskill lecture notes
- Nielsen,Chuang book
- Wilde, Quantum Shannon theory