Applied Quantum Information Science SoSe 2023

Lecturer: Dr. Peter P. Orth **Problem Set 1, 100 points**
Due date: Mo, 6/12/2023, 11:59 PM

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1. **Projective measurements** $(5 + 5 = 10$ points$)$
Consider a quantum system in the computational basis state $|0\rangle$, i.e., $Z|0\rangle = |0\rangle$.
   (a) Calculate the probability of obtaining the result $+1$ for a measurement of $\boldsymbol{v} \cdot \boldsymbol{\sigma}$, where $|\boldsymbol{v}| = 1$.
   (b) What is the state of the system after the measurement if $+1$ is obtained?

2. **Expectation values, measurements, and the Bloch sphere** $(4 \times 10 = 40$ points$)$
The following questions refer to the pure one-qubit state
$$|\psi(\theta, \phi)\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \ ,$$
which represents a point (θ, ϕ) on the Bloch sphere. Assume that we have access to quantum hardware sufficiently advanced to repeatedly prepare the state $|\psi(\theta, \phi)\rangle$ with no errors and perform measurements in X, Y, and Z bases.
   (a) Calculate the expectation values $\langle Z\rangle, \langle X\rangle$, and $\langle Y\rangle$ as functions of θ and ϕ. Using these expressions, calculate $\langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2$.
   (b) Calculate the standard deviations $\Delta(X)$ and $\Delta(Z)$ and prove that these expressions satisfy the Heisenberg uncertainty principle $\Delta(X)\Delta(Z) \geq |\langle\psi|[X, Z]|\psi\rangle|/2$ for any (θ, ϕ). [*Hint*: Square both sides of the Heisenberg uncertainty equation and subtract the right-hand side from both sides—this reduces the problem to showing that something is greater than or equal to 0.]
   (c) One might guess that any *two* of the expectation values $\langle Z\rangle, \langle X\rangle$, and $\langle Y\rangle$ are sufficient to determine (θ, ϕ), since we only need to fix two parameters. This is not quite true! Calculate θ and φ in terms of $\langle Z\rangle$ and $\langle X\rangle$ and show that in order to determine φ unambiguously we need to also know the *sign* of $\langle Y\rangle$.
   (d) The state $|\psi\rangle$ is measured 10^4 times in the Z basis, and the result $+1$ is obtained 3847 times. When $|\psi\rangle$ is measured 10^4 times in the X basis, and the result $+1$ is obtained 6523 times. A third set of measurements finds that $\langle Y\rangle > 0$. Calculate θ and ϕ based on this series of measurements.

3. **POVM measurements** $(10 + 10 = 20$ points$)$
Suppose Alice gives Bob a qubit prepared in one of two states, $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$. Bob's task is to find out which state he has.
   (a) Argue why Bob cannot determine which state he has with perfect reliability using projective measurements $P_m = |m\rangle\langle m|$, where $|m\rangle$ forms an orthonormal basis.
   (b) It turns out that Bob cannot determine the type of state he has with perfect reliability using any measurement scheme. However, using POVM measurements he can choose three non-orthogonal measurement operators M_m, $m = 1, 2, 3$, such that he will never misidentify the state. This works by constructing POVM operators in such a way that outcome $m = 1$ never occurs in state $|\psi_1\rangle$ and outcome $m = 2$ never occurs in state $|\psi_2\rangle$. By obtaining $m = 1$, he thus knows with certainty that the system was in state $|\psi_2\rangle$, and by obtaining $m = 2$, he knows it was $|\psi_1\rangle$. If he

obtains $m = 3$, he cannot tell which state it was, but at least he never misidentifies the state.

Construct the POVM measurement operators M_m and give the probabilities $p(m)$.

4. **Reduced density matrix and entropy** (5 + 5 + 10 + 10 = 30 points)

Consider the so-called Werner state

$$\rho(\lambda) = \lambda \frac{I}{4} + (1-\lambda) |\Psi^-\rangle \langle \Psi^-| \,,$$

where $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is one of the four Bell states.

(a) Explicitly state the density matrix in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

(b) Compute the von-Neumann entropy $S(\rho) = -\mathrm{Tr}\rho \ln \rho$ and plot the result. You can plot the result on the computer and either print it out or make a schematic drawing of the result by hand.

(c) Determine the von-Neumann entropy of the reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$, where you trace out the second qubit (system B).

(d) One can show that the state is separable (i.e. not entangled) if $1/\mathrm{Tr}(\rho^2) \geq 3$ [see K. Zyczkowski *et al.*, Phys. Rev. A **58**, 883 (1998)]. A state is separable if it can be expressed as a convex linear combination of product states $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$, where A, B refer to the two subsystems. Calculate $1/\mathrm{Tr}\rho^2$ and determine the value of λ that separates entangled states $\rho(\lambda)$ from separable ones.