Applied Quantum Information Science SoSe 2023

Lecturer: Dr. Peter P. Orth **Problem Set 2, 100 points**
Due date: Wed, 6/28/2023, 11:59 PM

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1. **Distance and fidelity of single-qubit states** (5 + 5 = 10 points)
Consider two general pure states of a single qubit $\psi_a(\theta_a, \phi_a) = \cos\frac{\theta_a}{2}|0\rangle + e^{i\phi_a}\sin\frac{\theta_a}{2}|1\rangle$ with $a = 1, 2$.

   (a) Calculate the trace distance $D(|\psi_1\rangle, |\psi_2\rangle)$ between the two states for general angles.

   (b) Calculate the fidelity $F(|\psi_1\rangle, |\psi_2\rangle)$ for general angles

2. **Schmidt decomposition** (10 + 10 = 20 points)
Perform the Schmidt decomposition of the pure state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\Big[|00\rangle + |01\rangle + |10\rangle\Big].$$

   (a) Give explicitly the matrices u, d, v that appear in the SVD of $a = udv^\dagger$. Then, state explictly the Schmidt basis states $|w_i\rangle_A$ and $|w_i\rangle_B$ in terms of the computational basis states $|i\rangle_A$ and $|i\rangle_B$.

   (b) Determine the von-Neumann entropy $S_A = -\text{Tr}_A\rho_A \ln\rho_A$ of the first qubit after tracing out the second one.

3. **Bell measurements** (3 × 10 = 30 points)
In this problem you'll demonstrate two ways to perform a measurement that distinguishes the four Bell states

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}},$$

where x and $y = 0, 1$ and where $\bar{0} = 1$ and $\bar{1} = 0$.

   (a) One way to distinguish the four Bell states is to apply a unitary circuit U' that maps $|\beta_{xy}\rangle$ to the four computational basis states $|xy\rangle$, i.e.,

$$U'|\beta_{xy}\rangle = |xy\rangle$$

   and then perform a standard measurement in the computational basis. This method is useful on real quantum devices. Find the circuit U'. [*Hint*: Consider the Bell-state preparation circuit presented in class.]

Another way to distinguish the Bell states is to find *commuting* observables M_1 and M_2 (i.e., $[M_1, M_2] = 0$) with eigenvalues ± 1 whose eigenstates are $|\beta_{xy}\rangle$. A projective measurement of these observables would yield unambiguous confirmation of which Bell state is being measured.

   (b) Show that the Pauli strings $M_1 = Z \otimes Z \equiv Z_1 Z_2$ and $M_2 = X \otimes X \equiv X_1 X_2$ each have eigenvalues ± 1 and satisfy $[M_1, M_2] = 0$.

   (c) Show that all four $|\beta_{xy}\rangle$ are eigenvectors of M_1 and M_2 and that no two $|\beta_{xy}\rangle$ have the same set of eigenvalues of M_1 and M_2.

4. Operator-sum representations (10 + 10 = 20 points)

In this problem, you will derive the operator-sum representation of a quantum channel starting from the unitary description of the system coupled to an environment.

(a) Suppose we have a single qubit principal system A, interacting with a single qubit environment A through the transform

$$U = P_0 \otimes I + P_1 \otimes X$$

where X is the usual Pauli matrix (acting on the environment), and $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum quantum channel $\mathcal{E}$ for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle_E$.

(b) Just as in the previous question, but now let

$$U = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X$$

Give the quantum channel for this process, in the operator-sum representation.

5. Gate fidelity (10 + 10 = 20 points)

(a) To measure the closeness of a noisy quantum operation $\mathcal{E}$ to a desired unitary operation U one defines the *gate fidelity*

$$F(U, \mathcal{E}) = \min_{|\psi\rangle} F\Big(U|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)\Big)$$

Suppose that the noisy quantum channel of a NOT (or X) gate is given by $\mathcal{E} = (1-p)X\rho X + pZ\rho Z$, find the gate fidelity as a function of p.

(b) Suppose U and V are unitary operators, and $\mathcal{E}$ and $\mathcal{F}$ are trace-preserving quantum operations meant to approximate U and V. Letting $d(\cdot,\cdot)$ be any metric on the space of density matrices satisfying $d(U\rho U^\dagger, U\sigma U^\dagger) = U(\rho, \sigma)$ for all density matrices ρ and σ and unitary U (one example is the angle $\arccos[F(\rho, \sigma)]$), define the corresponding error $E(U, \mathcal{E})$ by

$$E(U, \mathcal{E}) = \max_\rho d(U\rho U^\dagger, \mathcal{E}(\rho))$$

and show that $E(VU, \mathcal{F} \circ \mathcal{E}) \leq E(U, \mathcal{E}) + E(V, \mathcal{F})$. Thus, to perform a quantum computation with high fidelity it suffices to complete each step of the computation with high fidelity.