Saarland University — Department of Physics

Applied Quantum Information Science SoSe 2023

Lecturer: Prof. Dr. Peter P. Orth — **Problem Set 3, 100 points**

Due date: Wed, 7/12/2023, 11:59 PM

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1. Process and Pauli transfer matrices (10 + 10 = 20 points)

Suppose a projective measurement is performed on a single qubit in the X basis $\pm = \frac{1}{\sqrt{2}}\left[|0\rangle \pm |1\rangle\right]$. In the event that we are ignorant of the result of the measurement, the density matrix evolves according to the equation

$$\rho \to \mathcal{E}(\rho) = |+\rangle\langle+|\rho|+\rangle\langle+| + |-\rangle\langle-|\rho|-\rangle\langle-| \,. \qquad (1)$$

(a) Derive the process matrix χ in the Pauli basis for this quantum operation and check that the operation is CP.

(b) Derive the Pauli transfer matrix $(R_{\mathcal{E}})$ and calculate $|\rho'\rangle\rangle = (R_{\mathcal{E}})|\rho\rangle\rangle$ with $|\rho\rangle\rangle$ being the vectorized density matrix $\rho = \frac{1}{2}(I + \mathbf{r}\cdot\boldsymbol{\sigma})$ in the Pauli basis. Use your results to illustrate this transformation $\mathcal{E}(\rho)$ on the Bloch sphere by plotting the resulting set of points $\mathbf{r}' = (r'_x, r'_y, r'_z)$

2. Circuit models for amplitude and phase damping (10 + 10 = 20 points)

(a) Show that the circuit in the left figure below models the amplitude damping quantum operation, with $\sin^2(\theta/2) = p$.

(b) Show that the circuit in the right figure below can be used to model the phase damping quantum operation, provided θ is chosen appropriately.

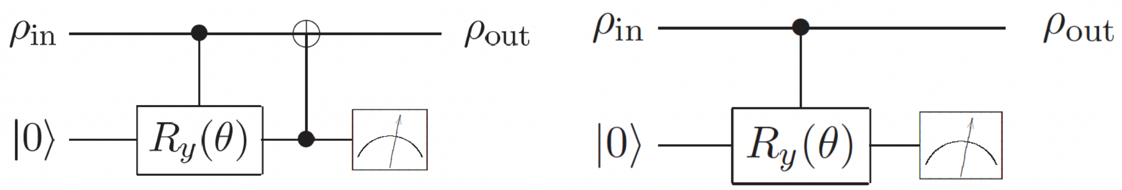

Figure 1: (Left) Circuit model for amplitude damping with $p = \sin^2(\theta/2)$. (Right) Circuit model for phase damping, provided θ is chosen appropriately.

3. TP maps are contractive (10 + 10 = 20 points)

(a) Consider the depolarizing channel introduced $\mathcal{E}(\rho) = (1-p)\rho + pI/2$. For arbitrary ρ and σ find the trace distance $D[\mathcal{E}(\rho), \mathcal{E}(\sigma)]$ using the Bloch representation $\rho = \frac{1}{2}(I + \mathbf{r}\cdot\boldsymbol{\sigma})$, and prove explicitly that the map $\mathcal{E}$ is strictly contractive, that is, $D[\mathcal{E}(\rho), \mathcal{E}(\sigma)] < D(\rho, \sigma)$.

(b) Show that the bit flip channel $M_0 = \sqrt{1-p}I$, $M_1 = \sqrt{p}X$ is a contractive channel $D[\mathcal{E}(\rho), \mathcal{E}(\sigma)] \leq D(\rho, \sigma)$ but not strictly contractive. Find the set of fixed points $\rho = \mathcal{E}(\rho)$ for the bit flip channel.

4. Quantum state tomography with noisy circuits (8 × 5 = 40 points)

In this question you will investigate how hardware imperfections affect ciruit calculations using a noise model proposed by Kandala et al. in Nature **549**, 242 (2017). This model consists of an amplitude damping channel $\mathcal{E}_{ad}[\rho] = \sum_{i=0}^{1} M_{ad,i}\rho M_{ad,i}^{\dagger}$ and a dephasing (or phase damping) channel $\mathcal{E}_{pd}[\rho] = \sum_{i=0}^{1} M_{pd,i}\rho M_{pd,i}^{\dagger}$. Both channels act on the qubit density matrix after each single-qubit or two-qubit gate operation. The Kraus operators are defined as follows:

$$M_{ad,0} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p_{ad}} \end{pmatrix}, M_{ad,1} = \begin{pmatrix} 0 & \sqrt{p_{ad}} \\ 0 & 0 \end{pmatrix},$$
$$M_{pd,0} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p_{pd}} \end{pmatrix}, M_{pd,1} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p_{pd}} \end{pmatrix}. \quad (2)$$

The error rates $p_{ad} = 1 - e^{-t_g/T_1}$ and $p_{pd} = 1 - e^{-2t_g/T_\phi}$ depend on the gate time t_g, qubit relaxation time T_1, and dephasing time $T_\phi = 2T_1T_2/(2T_1 - T_2)$, where T_2 is the qubit coherence time. Since t_g depends on the gate being performed, this noise model assumes a different error rate for each gate. For simplicity in our analysis, we assume a uniform single-qubit gate error rate $p_{ad,1q} = p_{pd,1q} \equiv p_1 = 10^{-4}$ and a uniform two-qubit error rate $p_{ad,2q} = p_{pd,2q} = p_2 = 10^{-2}$. These values closely match those of current hardware. Here, we will study a system of two noisy qubits.

(a) Define the combined channel $\mathcal{E} = \mathcal{E}_{dp} \circ \mathcal{E}_{ad}$ for two qubits using `qiskit.aer.noise`. You should use the `NoiseModel()` and `Quantum_Error()` classes, as described in this tutorial. Start by defining the two quantum channels $\mathcal{E}_{ad}$ and $\mathcal{E}_{pd}$ for single and two-qubit gates, respectively, using `qiskit.quantum_info.operators.channel.kraus`. Note that the two-qubit error channel is a simple tensor product of the one-qubit model. Finally, define a backend `AerSimulator(noise_model = your_noise_model)` that contains two noisy qubits.

(b) Derive the Choi process matrix in both the column vectorized basis `qiskit.quantum_info.operators.channel.choi` and the Pauli basis `qiskit.quantum_info.operators.channel.chi`. You can use the tools provided in `qiskit.quantum_info.operators.channel.transformations`. Show explicitly that the channel is CP.

(c) Derive the superoperator representation of the quantum channel (for one qubit) in both the column vectorized and the Pauli basis (which yields the Pauli transfer matrix). Show that the channel is TP. Is it also unital?

(d) Show explicitly by averaging over the Pauli group $\mathcal{P}$ that the Pauli transfer matrix becomes diagonal under Pauli twirling $\mathbb{T}_{\mathcal{P}}(\mathcal{E}) = \frac{1}{|\mathcal{P}|}\sum_{P\in\mathcal{P}} P^\dagger \mathcal{E} P$.

(e) Show explicitly by averaging over the Clifford group $\text{Clif}_{n=2}$ that the Pauli transfer matrix takes the form of depolarizing noise under Clifford twirling $\mathbb{T}_{\text{Clif}_n}(\mathcal{E}) = \frac{1}{|\text{Clif}_n|}\sum_{C\in\text{Clif}_n} C^\dagger \mathcal{E} C$. The Clifford group is generated by $\{H, S, CNOT\}$. Study both the single-qubit and the two-qubit error channels.

(f) Consider a circuit acting on two qubits that prepares the Bell state $|\beta_{00}\rangle \equiv |\Phi^+\rangle$. The goal of this question is to compare the noisy state ρ and the noiseless state ρ_0. First, compute both ρ and ρ_0 using an exact statevector simulation. You could use the superoperators to obtain ρ. Then compute the fidelity and the trace distance between ρ and ρ_0.

(g) Now perform quantum state tomography on the resulting two-qubit final state using circuits with N_{sh} circuit executions. Employ the Pauli measurement basis and the Pauli preparation basis discussed in class on each qubit $\{|0\rangle, |1\rangle, |+\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle], |-\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i\,|1\rangle]\}$. These states are called $|Zp\rangle$, $|Zm\rangle$, $|Xp\rangle$, $|Yp\rangle$ in `qiskit_experiments.library.tomography`. This defines the matrices A and B introduced in class. Then measure the matrix P from executing the circuits using a finite number of shots $N_{\text{sh}} = 2^{12}$. Obtain the Pauli transfer matrix using the quasi-inverse discussed in class. Does it obey the CPTP constraint?

(h) Now perform the same experiment using the tools in `qiskit_experiments.library.tomography`. Compare the results obtained with the minimal four Pauli basis preparation states `PauliPreparationBasis`, which should agree with your results obtained in the previous question, with results obtained with the overcomplete preparation basis `Pauli6PreparationBasis`.