Applied Quantum Information Science SoSe 2023

Lecturer: Prof. Dr. Peter P. Orth **Problem Set 4, 100 points**
Due date: Wed, 7/19/2023, 11:59 PM

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1. Quantum simulation (10 + 10 + 10 = 30 points)

(a) Construct a quantum circuit to simulate the Hamiltonian

$$H = X_1 Y_2 Z_3 \,, \tag{1}$$

performing the unitary transform $U(t) = e^{-iHt}$ for arbitrary t. Note that you do not need to use a Trotter decomposition in this case.

(b) Implement the circuit using Qiskit and compute the total Z magnetization $M_Z(t) = \frac{1}{3}\sum_{i=1}^{3}\langle Z_i(t)\rangle$ for 50 equally spaced points between $t = 0$ and $t = 10$ at discrete times $t_i = i/5$, $i = 0, \dots, 50$. Start from the initial state $|\psi(0)\rangle = |000\rangle$. Compare a statevector simulation with a simulation using $N_{\text{cir}} = 2^{12}$ circuit evaluations at each time point.

(c) Now consider additional terms in the Hamiltonian $H = \frac{1}{4}\sum_{i=1}^{3} X_i + \frac{1}{4}\sum_{i=1}^{2} Z_i Z_{i+1} + X_1 Y_2 Z_3$. Use a Trotter decomposition to compute the total Z magnetization $M_Z(t)$ between $t = 0$ and $t = 10$ at discrete times $t_i = i/5$, $i = 0, \dots, 50$. Perform the simulation for different Trotter step sizes $\Delta t = \{0.005, 0.05, 0.2\}$. Compare a statevector simulation with a simulation using $N_{\text{cir}} = 2^{12}$ circuit evaluations at each time point t_i.

2. Pauli group (5 + 5 = 10 points)
The three Pauli matrices and the identity form a group under multiplication for when we multiply two Pauli matrices we get another Pauli matrix up to a phase.

(a) Explain why the full one-qubit Pauli group $\mathcal{P}_1$ has 16 elements and explicitly write down all of them.

(b) Show that $\langle iI, X, Z\rangle$ is a generating set of this group.

3. Stabilizers ($4 \times 5 = 20$ points)
One says that a unitary S stabilizes the state $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$. One can use stabilizers to define vectors and subspaces of vectors, which is useful, for example, in quantum error correction.

(a) Show that the set of stabilizers of $|\psi\rangle$ forms a group (known as the stabilizer group).

(b) Which states are stabilized by the Pauli matrices X, Y, Z and which by $-X$, $-Y$, and $-Z$?

(c) Which states are stabilized by the identity I and which by $-I$?

(d) What are the stabilizer groups of the computational basis states, $|0\rangle$ and $|1\rangle$?

4. Clifford group ($4 \times 10 = 40$ points)
The Clifford group $\text{Clif}_{n=1}$ of a single qubit is generated by the Hadamard gate H and the phase gate S. For two qubits, one also needs to include the $CNOT$ gate.

(a) Show how one can implement the gate $S^\dagger$ using just H and S gates.

(b) Show how to implement the three Pauli operators X, Y, and Z using just H and S gates.

(c) Show that, under conjugation, Clifford gates $C \in \mathrm{Clif}_{n=1}$ map Pauli operators to Pauli operators: $CPC^\dagger = P'$ (modulo phase factors), where P and P' are two Pauli operators. In other words, the Clifford group is defined as the group of unitaries that normalize the Pauli group.

(d) Explain why any circuit composed only of the single qubit Clifford gates maps the set of Pauli eigenstates to the set of Pauli eigenstates.