(1)

$$\vec{v}\cdot\vec{\sigma} = v_1 X + v_2 Y + v_3 Z \quad , \quad \vec{v} = (v_1, v_2, v_3) =$$

$|\vec{v}| = 1$

(a)

$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos(\theta))$$

Explicitly diagonalizing $\vec{v}\cdot\vec{\sigma}$ yields

$$\lambda_1 = 1, \quad \vec{w}_1 = \frac{1}{N}(1+v_3, \; v_1 + i v_2) = \left(\cos\frac{\theta}{2}, \; e^{i\phi}\sin\frac{\theta}{2}\right)$$

$$\lambda_2 = -1, \quad \vec{w}_2 = \frac{1}{N}(1-v_3, \; -(v_1+iv_2)) = \left(\sin\frac{\theta}{2}, \; -e^{i\phi}\cos\frac{\theta}{2}\right)$$

with normalization $N = \sqrt{(1+v_3)^2 + v_1^2 + v_2^2} = \sqrt{2+2v_3} =$
$= \sqrt{2}\sqrt{1+v_3}$

$$|\psi_0(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$|\psi_1(\theta,\phi)\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle .$$

Check that $|\psi_0(\theta,\phi)\rangle$ is indeed $+1$ eigenstate of $\vec{v}\cdot\vec{\sigma}$:

$$\vec{v}\cdot\vec{\sigma}\,|\psi_0(\theta,\phi)\rangle = \Big[\sin\theta\cos\phi\, X + \sin\theta\sin\phi\, Y$$

$$+ \cos\theta\, Z\Big]\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right] =$$

$$= S_\theta C_\phi\left(C_{\frac{\theta}{2}}|1\rangle + e^{i\phi}S_{\frac{\theta}{2}}|0\rangle\right) + S_\theta S_\phi\left(iC_{\frac{\theta}{2}}|1\rangle - ie^{i\phi}S_{\frac{\theta}{2}}|0\rangle\right)$$

$$+ C_\theta \left[C_{\frac{\theta}{2}} |0\rangle - e^{i\phi} S_{\frac{\theta}{2}} |1\rangle \right] =$$

$$= |0\rangle \left[\underbrace{S_\theta C_\phi S_{\frac{\theta}{2}} e^{i\phi} - i S_\theta S_\phi e^{i\phi} S_{\frac{\theta}{2}}}_{= S_\theta S_{\frac{\theta}{2}} e^{-i\phi} e^{i\phi} = S_\theta S_{\frac{\theta}{2}}} + C_\theta C_{\frac{\theta}{2}} \right]$$

$$+ |1\rangle \left[\underbrace{S_\theta C_\phi C_{\frac{\theta}{2}} + i S_\theta S_\phi C_{\frac{\theta}{2}}}_{= S_\theta C_{\frac{\theta}{2}} e^{i\phi}} - C_\theta S_{\frac{\theta}{2}} e^{i\phi} \right] =$$

$$= |0\rangle \left[S_\theta S_{\frac{\theta}{2}} + C_\theta C_{\frac{\theta}{2}} \right] + |1\rangle e^{i\phi} \left[S_\theta C_{\frac{\theta}{2}} - C_\theta S_{\frac{\theta}{2}} \right]$$

Now use $S_\theta = 2 S_{\frac{\theta}{2}} C_{\frac{\theta}{2}}$, $C_\theta = C^2_{\frac{\theta}{2}} - S^2_{\frac{\theta}{2}}$.

$$= |0\rangle \left[C_{\frac{\theta}{2}} \left(\underbrace{2 S^2_{\frac{\theta}{2}} + C^2_{\frac{\theta}{2}} - S^2_{\frac{\theta}{2}}}_{= C^2_{\frac{\theta}{2}} + S^2_{\frac{\theta}{2}} = 1} \right) \right] + |1\rangle e^{i\phi} \left[S_{\frac{\theta}{2}} \left(\underbrace{2 C^2_{\frac{\theta}{2}} - C^2_{\frac{\theta}{2}} + S^2_{\frac{\theta}{2}}}_{=1} \right) \right]$$

$$= \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle \qquad \square$$

Projector onto $\lambda = 1$ eigenspace thus

$$P_{\vec{v},0} = |\psi_0(\theta,\phi)\rangle\langle\psi_0(\theta,\phi)| =$$

$$= \cos^2\left(\frac{\theta}{2}\right)|0\rangle\langle 0| + \sin^2\left(\frac{\theta}{2}\right)|1\rangle\langle 1|$$

$$+ \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left[e^{-i\phi}\underbrace{|0\rangle\langle 1|}_{=\sigma^+ = \frac{1}{2}(X+iY)} + e^{i\phi}\overbrace{|1\rangle\langle 0|}^{=\sigma^- = \frac{1}{2}(X-iY)}\right].$$

We used

$$|\psi_0(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle.$$

Now, we can rewrite this as

$$P_{\vec{v},0} = \left(\cos^2\left(\frac{\theta}{2}\right) - \frac{1}{2} + \frac{1}{2}\right)|0\rangle\langle 0|$$

$$+\left(\underbrace{\sin^2\left(\frac{\theta}{2}\right) - \frac{1}{2}}_{=\frac{1}{2}-\cos^2\frac{\theta}{2}} + \frac{1}{2}\right)|1\rangle\langle 1|$$

$$+ \frac{1}{2}\underbrace{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}_{=\sin\theta}\left[\cos\phi\, X + \sin\phi\, Y\right]$$

$$= \frac{I}{2} + \overbrace{\left(\cos^2\frac{\theta}{2} - \frac{1}{2}\right)}^{=\frac{1}{2}\left(2\cos^2\frac{\theta}{2} - 1\right) = \frac{1}{2}\cos\theta}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$

$$+ \frac{1}{2}\sin\theta\left[\cos\phi\, X + \sin\phi\, Y\right]$$

$$\Rightarrow \boxed{P_{\vec{v},0} = \frac{1}{2}\left[I + \cos\theta\, Z + \sin\theta\cos\phi\, X + \sin\theta\sin\phi\, Y\right] = \frac{1}{2}\left[I + \vec{v}\cdot\vec{\sigma}\right].}$$

The probability to find +1 eigenvalue in state $|0\rangle$ is therefore

$$\boxed{p(\vec{v},0) = \mathrm{Tr}\left[P_{\vec{v},0}\,|0\rangle\langle 0|\right] = \frac{1}{2}(1+\cos\theta).}$$

(b)

After the measurement the state of the system is

$$\rho_f = \frac{P_{\vec{v},0}\,|0\rangle\langle 0|\,P_{\vec{v},0}}{p(\vec{v},0)} = \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma}).$$

or

$$|\psi_f\rangle = |\psi_0(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle.$$

(2) $|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

$\Rightarrow \rho(\theta,\phi) = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$

(a)

$\langle Z\rangle = \langle\psi(\theta,\phi)|\, Z\, |\psi(\theta,\phi)\rangle =$

$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$

$= \mathrm{Tr}[Z\rho] = \frac{1}{2} r_3 \,\mathrm{Tr}[I] = r_3 = \cos\theta.$

$\langle X\rangle = \mathrm{Tr}[X\rho] = \frac{1}{2} r_1 \,\mathrm{Tr}[I] = r_1 = \sin\theta\cos\phi$

$\langle Y\rangle = \mathrm{Tr}[Y\rho] = \sin\theta\sin\phi.$

$\Rightarrow \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 1.$

$X^2 = I$

(b) $\Delta(X) = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{1 - \sin^2\theta\cos^2\phi}$

$\Delta(Z) = \sqrt{1 - \cos^2\theta}$

Show Heisenberg uncertainty relation: $\sigma_a \sigma_b = i\sigma_c$ (cyclic)

$$\Delta(X)\,\Delta(Z) \geq \frac{1}{2}\left|\langle\psi|\underbrace{[X,Z]}_{=-2iY}|\psi\rangle\right| = \langle\psi|Y|\psi\rangle$$

Squaring both sides yields

$$(1-\sin^2\theta\cos^2\phi)(1-\cos^2\theta) \geq \sin^2\theta\sin^2\phi$$

$$\Leftrightarrow 1 - \underline{\sin^2\theta\cos^2\phi} - \cos^2\theta + \sin^2\theta\cos^2\phi\cos^2\theta$$
$$- \underline{\sin^2\theta\sin^2\phi} \geq 0$$

$$\Leftrightarrow 1 \underbrace{-\sin^2\theta - \cos^2\theta}_{=-1} + \sin^2\theta\cos^2\phi\cos^2\theta \geq 0$$

$$\Leftrightarrow \sin^2\theta\cos^2\theta\cos^2\phi \geq 0$$

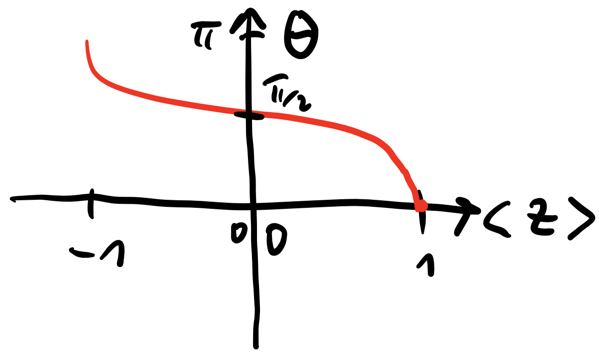
LHS is product of 3 non-negative numbers.

Bound is satisfied for $\cos\phi = 0 \Rightarrow \phi = 0, \pi$.

$\Rightarrow \langle Y\rangle = 0$ then.

(c)

$$\langle X\rangle = \sin\theta\cos\phi$$
$$\langle Y\rangle = \sin\theta\sin\phi$$
$$\langle Z\rangle = \cos\theta$$

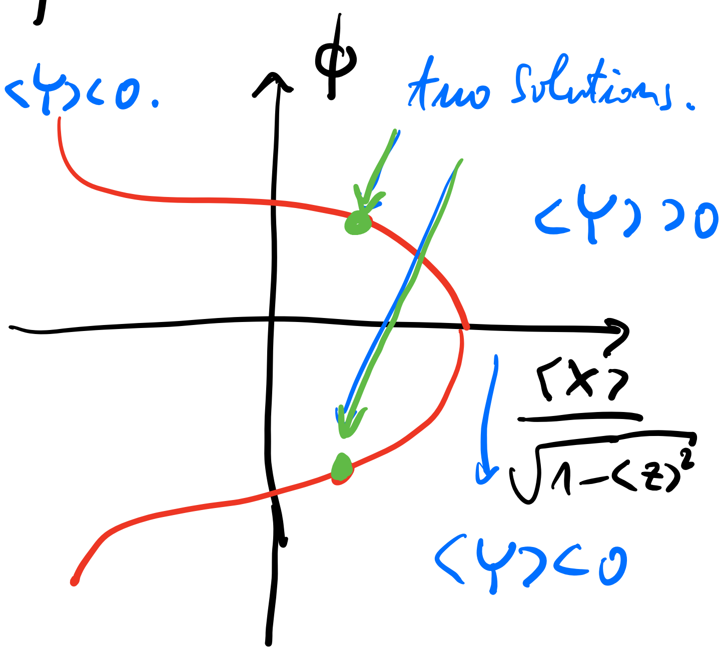


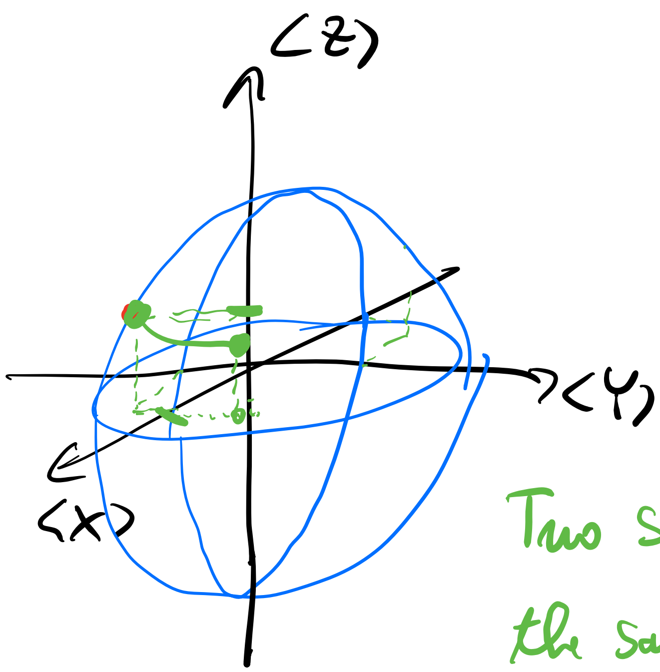
$$\Rightarrow \theta = \operatorname{acos}\langle Z\rangle$$

$$\Rightarrow \phi = \pm\operatorname{acos}\left(\frac{\langle X\rangle}{\sqrt{1-\langle Z\rangle^2}}\right)$$ up to a sign

+ if $\langle Y\rangle>0$, − if $\langle Y\rangle<0$.

$$\phi = \operatorname{atan}\frac{\langle Y\rangle}{\langle X\rangle}$$





Two solutions for ϕ with the same value of $\langle X\rangle$ and $\langle Z\rangle$, unless $\langle Y\rangle=0$.

(d)

$M = 10^4$ measurements in computational basis

$\hat{=}$ Z basis

Find $M_0 = 3847$, eigenvalue $+1 = \lambda_0 \Rightarrow m_0 = \frac{M_0}{M}$

and $M_1 = 1 - M_0$, eigenvalue $-1 = \lambda_1 \Rightarrow m_1 = \frac{M_1}{M}$

$$\Rightarrow \langle Z \rangle = m_0 - m_1 = 2m_0 - 1 = -0.2306$$

When measured in X basis, we find

$$M_+ = 6523, \quad M_- = M - M_+$$

$$\Rightarrow \langle X \rangle = m_+ - m_- = 2m_+ - 1 = 0.3046$$

Since $\langle Y \rangle > 0 \Rightarrow \phi \geq 0$ and

$$\theta = \operatorname{acos} \langle Z \rangle = 1.803 \hat{=} 103.3°$$

$$\phi = + \operatorname{acos} \frac{\langle X \rangle}{\sqrt{1 - \langle Z \rangle^2}} = 1.252 \hat{=} 71.76°.$$

③

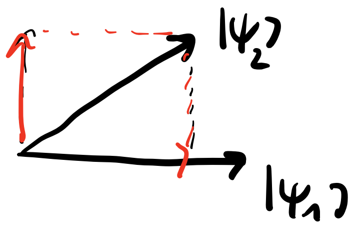
ⓐ

$$|\psi_1\rangle = |0\rangle \text{ and } |\psi_2\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

are not orthogonal, i.e., have a finite overlap $\langle\psi_1|\psi_2\rangle \neq 0$.

Orthonormal projection operators $|i\rangle\langle i| = P_i$ fulfill

$P_i P_j = P_i \delta_{ij}$. We can write $|\psi_1\rangle = \cos\alpha\, P_1 + \sin\alpha\, P_2$



$\Rightarrow |\psi_2\rangle = \cos\beta\, P_1 + \sin\beta\, P_2$

with $\langle\psi_1|\psi_2\rangle = \frac{1}{\sqrt{2}} = \cos\alpha\cos\beta + \sin\alpha\sin\beta =$

$= \cos(\alpha-\beta)$

$\Rightarrow \alpha - \beta = \frac{\pi}{4} = 45°.$

$\Rightarrow$ In state $|\psi_1\rangle$: $p_1 = \langle\psi_1|P_1|\psi_1\rangle = \cos^2\alpha$

$p_2 = \sin^2\alpha$

In state $|\psi_2\rangle$: $q_1 = \langle\psi_2|P_1|\psi_2\rangle = \cos^2\beta$

$q_2 = \sin^2\beta$

If we choose β such that outcome 1 never occurs in state $|\psi_2\rangle$, i.e., $\beta = \frac{\pi}{2}$, then $\alpha = \frac{3\pi}{4}$ and

$$\Rightarrow \; p_1 = \cos^2\alpha = \frac{1}{2}$$

$$p_2 = \sin^2\alpha = \frac{1}{2}$$

$\Rightarrow$ there is thus a finite probability to observe outcome 2 even in state 1.

We thus can misidentify the state when outcome 2 occurs as $p_2 = \frac{1}{2} \neq 0$ and $q_2 = 1 \neq 0$ then.

When outcome 1 occurs, Bob knows that his state was $|\psi_1\rangle$.

The same is true for any other choice of angles α and β.

There is always a nonzero probability to observe the same outcome (for at least one of the possible outcomes) in both states (due to the finite overlap $\langle\psi_1|\psi_2\rangle \neq 0$).

(b) POVM

$$E_1 = c\left(I - |\psi_1\rangle\langle\psi_1|\right) = c\left(I - |0\rangle\langle 0|\right) = c\,|1\rangle\langle 1|$$

$$E_2 = c\left(I - |\psi_2\rangle\langle\psi_2|\right) = c\left(I - \frac{1}{2}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right]\right) =$$

$$= c\,\frac{\left(|0\rangle - |1\rangle\right)\left(\langle 0| - \langle 1|\right)}{2}$$

$$E_3 = I - E_1 - E_2 .$$

We require that all 3 operators are positive $E_a \geq 0$.

→ this sets an upper bound for constant c.

Eigenvalues of E_3 are $\lambda_{3,\pm} = \frac{1}{2}\left[2 - 2c \pm \sqrt{2}\,c\right]$.

Now, solving for $\lambda_{3,-}(c) = 0 \Rightarrow C_{max} = \frac{2}{2+\sqrt{2}} = \frac{\sqrt{2}}{1+\sqrt{2}}$.

Thus, POVM is

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}\,|1\rangle\langle 1|$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\,\frac{\left(|0\rangle - |1\rangle\right)\left(\langle 0| - \langle 1|\right)}{2}$$

$$E_3 = I - E_1 - E_2$$

Thus, $m=1$ never occurs in state $|\psi_1\rangle$ as

$$\langle\psi_1|E_1|\psi_1\rangle = 0.$$

Similarly, $m=2$ never occurs in state $|\psi_2\rangle$.

<u>All probabilities:</u>

In state $|\psi_1\rangle$:

$$p_1 = \langle\psi_1|E_1|\psi_1\rangle = 0$$

$$p_2 = \langle\psi_1|E_2|\psi_1\rangle = \frac{\sqrt{2}}{1+\sqrt{2}}\cdot\frac{1}{2} = \frac{1}{2+\sqrt{2}} \simeq 0.293$$

$$p_3 = 1 - p_1 - p_2 = \frac{1}{\sqrt{2}} = 0.707$$

In state $|\psi_2\rangle$:

$$q_1 = \langle\psi_2|E_1|\psi_2\rangle = 0.293$$

$$q_2 = 0$$

$$q_3 = 0.707$$

Thus, in 71% of the measurements, Bob cannot tell which state he has, but in the other 29% of the measurements he knows. When he finds $m=1$, it was $|\psi_2\rangle$ and if he finds $m=2$, it was $|\psi_1\rangle$.

If he finds $m=3$, he does not know which state he found.

Distance between states and optimal POVM to distinguish states:

We learned in lecture 4 that

- $D(\rho,\sigma) = \max\limits_{\{E_m\}} D(p_m, q_m)$

trace distance

Here, $D(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}|\rho-\sigma| = \frac{1}{2}\mathrm{Tr}\left|\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| =$

$$= \frac{1}{2} \mathrm{Tr} \sqrt{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}} = \frac{1}{\sqrt{2}}.$$

For our POVM:

$$D(p_m, q_m) = \frac{1}{2} \sum_{m=1}^{3} |p_m - q_m| =$$

$$= \frac{1}{2}(2 \cdot 0.293) = 0.293 = \frac{1}{2+\sqrt{2}}.$$

So this POVM is not the optimal choice for discriminating the states based on the measurement probability distributions that Bob would obtain when measuring many instances of the states. However, when choosing this POVM, Bob never misidentifies the state that Alice sent him.

By the way:

For the projective measurement POVM, we found

$$p_1 = 0, \quad p_2 = 1$$

$$q_1 = q_2 = \frac{1}{2}$$

$$\Rightarrow D(p_m, q_m) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$$

This is also not the optimal POVM to distinguish the

States $|\psi_1\rangle$ and $|\psi_2\rangle$ based on $\{p_m\}$ and $\{q_m\}$.

Going back to the section on projective measurements above, we found

$$p_1 = \cos^2\alpha \,, \quad p_2 = \sin^2\alpha$$

$$q_1 = \cos^2\beta \,, \quad q_2 = \sin^2\beta$$

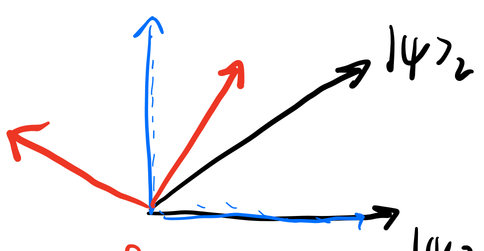
with $\alpha - \beta = \frac{\pi}{4}$.

The resulting $D(p_m, q_m) = \frac{1}{2}\sum_m |p_m - q_m|$ as a function of α is shown below.

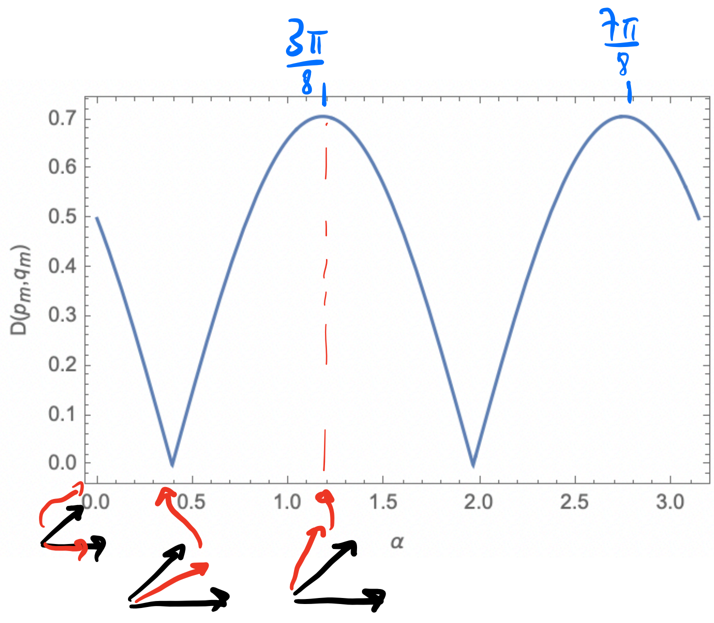
We find the maximal value

$$D = \frac{1}{\sqrt{2}} = D(\rho_1, \rho_2)$$

at $\alpha = \frac{3\pi}{8}$.



POVM for D = max., $\alpha = \frac{3\pi}{8}$



④

Werner state (mixture):

separable fully mixed state $\frac{I_1}{2} \otimes \frac{I_2}{2}$

$$\rho(\lambda) = \lambda \frac{I}{4} + (1-\lambda) \, |\Psi^-\rangle\langle\Psi^-|$$

max. entangled state

where $|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left[|01\rangle - |10\rangle \right]$

Convex combination of separable state $\frac{I}{4}$ and entangled state $|\psi^-\rangle\langle\psi^-|$.

ⓐ Density matrix

$$|\Psi^-\rangle\langle\Psi^-| = \frac{1}{2} \Big[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10| \Big]$$

$$\Rightarrow \rho = \begin{array}{c} \begin{pmatrix} \frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} - \frac{\lambda}{4} & -\frac{1}{2} + \frac{\lambda}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{\lambda}{2} & \frac{1}{2} - \frac{\lambda}{4} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{4} \end{pmatrix} \\ \begin{array}{cccc} \;|00\rangle\; & \;|01\rangle\; & \;|10\rangle\; & \;|11\rangle\; \end{array} \end{array}$$

$S(\lambda=0)=0$

$\lambda = 0$: $p_i \in \{1, 0, 0, 0\}$, pure state $|\psi^-\rangle\langle\psi^-|$

$\lambda = \frac{2}{3}$: $p_i \in \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}\}$, $S(\lambda=\frac{2}{3}) \approx 1.24 =$
$= \frac{1}{2}\ln 6 + \frac{1}{2}\ln 2$.

$\lambda = 1$: $p_i \in \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$: fully mixed state $\frac{I}{4}$.
$S(\lambda=1) = \ln 4 = 2\ln 2$.

$\lambda = \frac{4}{3}$: $p_i \in \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\}$: $S(\lambda=\frac{4}{3}) = \ln 3$

(b)

von-Neumann entropy

$$S = -\mathrm{Tr}\, \rho \ln \rho = -\sum_{i=1}^{4} \lambda_i \ln \lambda_i ,$$

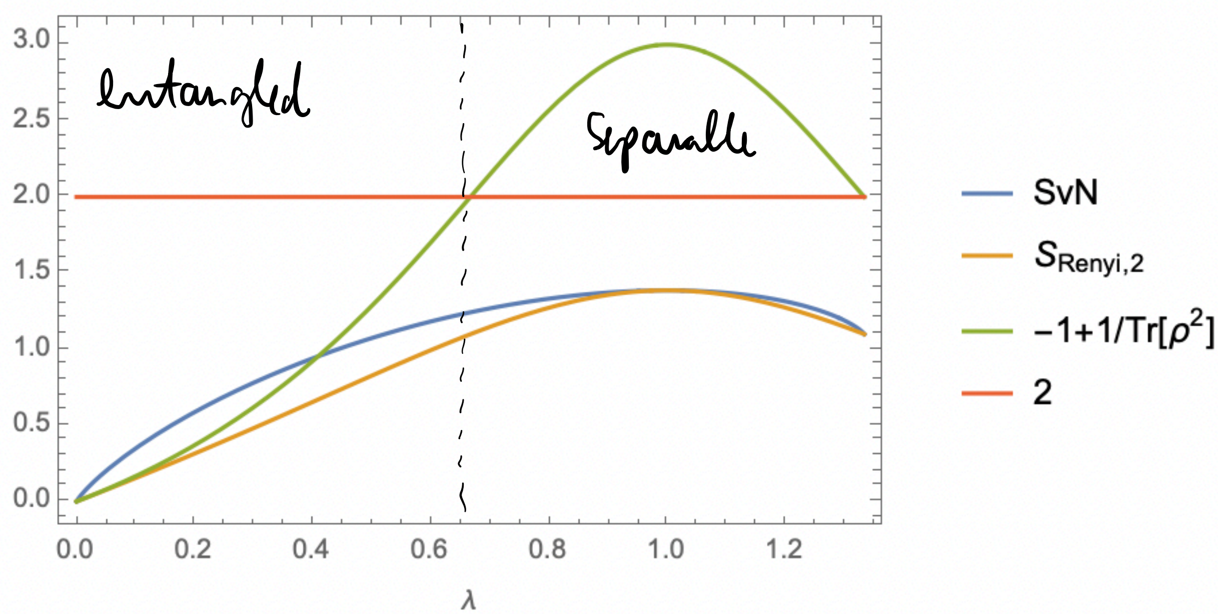
where λ_i are the eigenvalues of the Werner state:

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{\lambda}{4}$$

$$\lambda_4 = 1 - \frac{3}{4}\lambda \Rightarrow \text{positive for } 0 \leq \lambda \leq \frac{4}{3}.$$

Thus,

$$\boxed{S = -\left(1-\frac{3}{4}\lambda\right) \ln\left(1-\frac{3}{4}\lambda\right) - \frac{3}{4}\lambda \ln\frac{\lambda}{4}}$$



ⓒ von-Neumann entropy of reduced DM of first qubit:

$$\rho_A = \mathrm{Tr}_B\, \rho \;\rightarrow\; S_A = -\rho_A \ln \rho_A = -\sum_{i=1}^{2} \lambda_{A,i} \ln \lambda_{A,i}$$

$$\rho = \begin{pmatrix} \frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2}-\frac{\lambda}{4} & -\frac{1}{2}+\frac{\lambda}{2} & 0 \\ 0 & -\frac{1}{2}+\frac{\lambda}{2} & \frac{1}{2}-\frac{\lambda}{4} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{4} \end{pmatrix}$$

$$\begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix}$$

$$\Rightarrow \quad \rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1/2 \end{pmatrix}$$

Fully mixed state for all values of λ.

Thus,

$$\boxed{S_A = -2 \cdot \frac{1}{2} \ln \frac{1}{2} = \ln 2}$$

(d)

$$\frac{1}{\mathrm{Tr}\,\rho^2} = \frac{1}{\sum_{i=1}^{4} \lambda_i^2}$$

The green curve in the plot above shows

$$\frac{1}{\mathrm{Tr}\,\rho^2} - 1 > 2 \Rightarrow \text{state is separable}$$

$$\frac{1}{\mathrm{Tr}\,\rho^2} - 1 < 2 \Rightarrow \text{state is entangled.}$$

We find the state to be entangled for $0 \le \lambda < \frac{2}{3}$ and to be separable for $\frac{2}{3} < \lambda \le \frac{4}{3}$.

The critical value of λ that separates the two regimes is thus $\lambda_c = \frac{2}{3}$.