1

$$\psi_a(\theta_a, \phi_a) = \cos\frac{\theta_a}{2}|0\rangle + e^{i\phi_a}\sin\frac{\theta_a}{2}|1\rangle, \quad a = 1,2$$

a Trace distance

$$D(|\psi_1\rangle, |\psi_2\rangle) = \frac{1}{2}\mathrm{Tr}\,|\rho_1 - \rho_2|$$

Here: $\rho_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{2}(I + \vec{r}_1\cdot\vec{\sigma})$

$\rho_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{2}(I + r_2\cdot\vec{\sigma})$

with $\vec{r}_a = (\sin\theta_a\cos\phi_a, \sin\theta_a\sin\phi_a, \cos\theta_a)$

$\Rightarrow$

$|A| = \sqrt{A^\dagger A}$

$$D = \frac{1}{2}\cdot\frac{1}{2}\mathrm{Tr}\,|(\vec{r}_1 - \vec{r}_2)\cdot\vec{\sigma}| =$$

$$= \frac{1}{4}\mathrm{Tr}\begin{pmatrix}\sqrt{2}\sqrt{1-\cos\theta_1\cos\theta_2 - \cos(\phi_1-\phi_2)\sin\theta_1\sin\theta_2}, & 0 \\ 0, & \text{same as (11) element}\end{pmatrix}$$

explicit calculation of $A^\dagger A$ using MMA

$$= \frac{1}{4}\mathrm{Tr}\begin{pmatrix}|\vec{r}_1-\vec{r}_2| & 0 \\ 0 & |\vec{r}_1-\vec{r}_2|\end{pmatrix} =$$

$$= \frac{1}{4}\,2|\vec{r}_1-\vec{r}_2| = \frac{1}{2}|\vec{r}_1-\vec{r}_2|.$$

(b)

$$F(|\psi_1\rangle, |\psi_2\rangle) = \mathrm{Tr}\sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}} =$$

$$= \mathrm{Tr}\sqrt{|\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_1\rangle\langle\psi_1|} =$$

$$= \mathrm{Tr}\sqrt{|\langle\psi_1|\psi_2\rangle|^2\,|\psi_1\rangle\langle\psi_1|} =$$

$$= |\langle\psi_1|\psi_2\rangle| =$$

$$= \Big|\left(\cos\frac{\theta_1}{2}\langle 0| + e^{-i\phi_1}\sin\frac{\theta_1}{2}\langle 1|\right)\cdot$$

$$\left(\cos\frac{\theta_2}{2}|0\rangle + e^{i\phi_2}\sin\frac{\theta_2}{2}|1\rangle\right)\Big| =$$

$$= \left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \underbrace{e^{-i(\phi_1-\phi_2)}}_{=\cos(\phi_1-\phi_2)-i\sin(\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|$$

Note it holds that

$$D(|\psi_1\rangle, |\psi_2\rangle) = \sqrt{1 - F(|\psi_1\rangle, |\psi_2\rangle)^2}.$$

②

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left[\,|00\rangle + |01\rangle + |10\rangle\,\right]$$

Schmidt decomposition

Write as

$$a = u\,d\,v$$

$$|\psi\rangle = \sum_{j,k} a_{jk}\,|j\rangle|k\rangle = \sum_{i,j,k} u_{ji}\,d_{ii}\,v_{ik}\,|j\rangle|k\rangle$$

$$= \sum_i d_{ii}\,|\tilde{i}\rangle_A|\tilde{i}\rangle_B$$

$$\Rightarrow (a_{jk}) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

with $|\tilde{i}\rangle_A = \sum_j u_{ji}\,|j\rangle_A$

$|\tilde{i}\rangle_B = \sum_k v_{ki}\,|k\rangle_B$.

Now perform SVD decomposition using MMA:

$$a = u\,d\,v = \sum_j u_{ij}\,d_{jj}\,v_{jk}$$

SVD: $a = u\,d\,v^{+} \Rightarrow \lambda_0 = \sqrt{\frac{1}{6}(3+\sqrt{5})}$

$$\lambda_1 = \sqrt{\frac{1}{6}(3-\sqrt{5})}$$

$$u = \begin{pmatrix} 0.85 & 0.53 \\ 0.53 & -0.85 \end{pmatrix} \Rightarrow \begin{aligned} |i=0\rangle_A &= 0.85\,|0\rangle_A + 0.53\,|1\rangle_A \\ |i=1\rangle_A &= 0.53\,|0\rangle_A - 0.85\,|1\rangle_A \end{aligned}$$

$$V = \begin{pmatrix} 0.85 & -0.53 \\ 0.53 & 0.85 \end{pmatrix} \qquad \begin{aligned} |i=0\rangle_B &= 0.85\,|\overset{V}{0}\rangle_B + 0.53\,|1\rangle_B \\ |i=1\rangle_B &= -0.53\,|0\rangle_B + 0.85\,|1\rangle_B \end{aligned}$$

$$\Rightarrow |\psi\rangle = \sqrt{\tfrac{1}{6}(3+\sqrt{5})}\;|i=0\rangle_A\,|i=0\rangle_B$$

$$+\sqrt{\tfrac{1}{6}(3-\sqrt{5})}\;|i=1\rangle_A\,|i=1\rangle_B$$

<u>Check</u>: $0.85 = a,\ 0.53 = b$

$$\lambda_0\left[a^2|00\rangle + b^2|11\rangle + ab\,|01\rangle + ab\,|10\rangle\right]$$

$$+\lambda_1\left[-b^2|00\rangle - a^2|11\rangle + ab\,|01\rangle + ab\,|10\rangle\right] =$$

$$= \underbrace{(\lambda_0 a^2 - \lambda_1 b^2)}_{=1/\sqrt{3}}\,|00\rangle + \underbrace{(\lambda_0 b^2 - \lambda_1 a^2)}_{=0}\,|11\rangle$$

$$+\underbrace{(\lambda_0 + \lambda_1)\,ab}_{=1/\sqrt{3}}\,\big(|01\rangle + |10\rangle\big) \qquad \checkmark$$

(b) von-Neumann entropy after tracing out second qubit is given by Schmidt eigenvalues: $S = -\sum_i \lambda_i^2 \ln \lambda_i^2 = 0.38$.

③

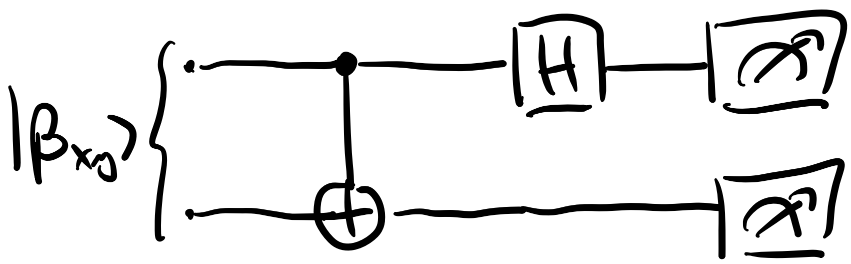
4 Bell states

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}}$$

with $x, y \in \{0,1\}$ and $\bar{0} = 1$, $\bar{1} = 0$.

ⓐ

Circuit U' that maps $|\beta_{xy}\rangle \xmapsto{U'} |xy\rangle$.



This circuit performs a measurement in the Bell basis, i.e, it maps

$|\beta_{00}\rangle \equiv |\phi^+\rangle \to |00\rangle$

$|\beta_{10}\rangle \equiv |\phi^-\rangle \to |10\rangle$

$|\beta_{01}\rangle \equiv |\Psi^+\rangle \to |01\rangle$

$|\beta_{11}\rangle \equiv |\Psi^-\rangle \to |11\rangle$

Here:

$$|\beta_{00}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$$|\beta_{10}\rangle = |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$

$$|\beta_{01}\rangle = |\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$

$$|\beta_{11}\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$$

Explicitly:

$$|\beta_{00}\rangle \equiv |\Phi^+\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|00\rangle + |10\rangle\right] \xrightarrow{H} |00\rangle$$

$$|\beta_{10}\rangle \equiv |\Phi^-\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|00\rangle - |10\rangle\right] \xrightarrow{H} |10\rangle$$

$$|\beta_{01}\rangle \equiv |\Psi^+\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|01\rangle + |11\rangle\right] \xrightarrow{H} |01\rangle$$

$$|\beta_{11}\rangle \equiv |\Psi^-\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left[|01\rangle - |11\rangle\right] \xrightarrow{H} |11\rangle$$

(b) Pauli strings $M_1 = Z_1 Z_2$, $M_2 = X_1 X_2$.

Eigenvalues of M_1:

$$M_1 = Z_1 Z_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$\Rightarrow$ eigenvalues ± 1.

$$M_2 = X_1 X_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Diagonalization yields eigenvalues $\lambda_{1,2} = 1$; $\lambda_{3,4} = -1$.

Compute $[M_1, M_2] = [Z_1 Z_2, X_1 X_2] =$

$$= Z_1 Z_2 X_1 X_2 - X_1 X_2 Z_1 Z_2 =$$

$$= Y_1 Y_2 - (-Y_1)(-Y_2) = 0 .$$

(c)

$$M_1 |\beta_{00}\rangle = Z_1 Z_1 | \beta_{00}\rangle =$$

$$= Z_1 Z_2 \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle \right] =$$

$$= |\beta_{00}\rangle$$

$$M_2 |\beta_{00}\rangle = X_1 X_2 \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle \right] =$$

$$= \frac{1}{\sqrt{2}}\left[|11\rangle + |00\rangle \right] =$$

$$= |\beta_{00}\rangle$$

<u>For $|\beta_{10}\rangle$:</u>

$$M_1 |\beta_{10}\rangle = Z_1 Z_1 \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle \right] =$$

$$= |\beta_{10}\rangle$$

$$M_2 |\beta_{10}\rangle = \frac{1}{\sqrt{2}}\left[|11\rangle - |00\rangle \right] = - |\beta_{10}\rangle .$$

For $|\beta_{01}\rangle$:

$$M_1 |\beta_{01}\rangle = Z_1 Z_2 \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle \right] =$$

$$= - |\beta_{01}\rangle$$

$$M_2 |\beta_{01}\rangle = |\beta_{01}\rangle .$$

For $|\beta_{11}\rangle$:

$$M_1 |\beta_{11}\rangle = Z_1 Z_2 \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle \right] =$$

$$= - |\beta_{11}\rangle$$

$$M_2 |\beta_{11}\rangle = - |\beta_{11}\rangle .$$

Table of eigenvalues of M_1 and M_2:

	M_1	M_2
$\|\beta_{00}\rangle$	1	1
$\|\beta_{01}\rangle$	-1	1
$\|\beta_{10}\rangle$	1	-1
$\|\beta_{11}\rangle$	-1	-1

Unique combinations $\Rightarrow$ Eigenvalues of M_1 and M_2 completely specify $|\beta_{xy}\rangle$.

④

Single qubit A + single qubit environment E

ⓐ $\mathcal{U} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$

$$\Rightarrow \quad \mathcal{U} : |\psi\rangle_A |0\rangle_E \longrightarrow \sum_a M_a |\psi\rangle |a\rangle_E$$

Kraus operators $\quad M_a = {}_E\langle a| \mathcal{U} |0\rangle_E$

here: $M_0 = P_0 = |0\rangle\langle 0|$

$M_1 = P_1 = |1\rangle\langle 1|$

Tracing out the environment yields quantum channel $\mathcal{E}$:

$$\mathcal{E} = \sum_{a=0}^{1} M_a \rho M_a^\dagger =$$

$$= P_0 \rho P_0^\dagger + P_1 \rho P_1^\dagger = \begin{pmatrix} \langle 0|\rho|0\rangle & 0 \\ 0 & \langle 1|\rho|1\rangle \end{pmatrix}$$

Find canonical Kraus operators via χ matrix.

[Turns out these are already the canonical ones $\mathrm{Tr}[M_a M_b] = \delta_{ab}$.)

Expand $M_a = \sum_b C_{ab} P_b$

$\Rightarrow \quad M_0 = \frac{1}{2}(I + Z)\,, \quad M_1 = \frac{1}{2}(I - Z)$

$$\Rightarrow \quad \mathcal{E} = \sum_{a,b} \chi_{ab}\, P_a\, \rho\, P_b =$$

$$= \frac{1}{4}\Big[(I+Z)\rho(I+Z) + (I-Z)\rho(I-Z) \Big]$$

with
$$\chi_{ab} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ eigenvectors $(1,0,0,0)^T$, $(0,0,0,1)^T$

$\Rightarrow$ unvectorize yields again $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$.

Can also use

$\tilde{M}_0 = \frac{I}{\sqrt{2}}$ and $\tilde{M}_1 = \frac{Z}{\sqrt{2}}$ as $M_0 = |0\rangle\langle 0| = \frac{1}{\sqrt{2}}[\tilde{M}_0 + \tilde{M}_1]$

$M_1 = \frac{1}{\sqrt{2}}[\tilde{M}_0 - \tilde{M}_1]$

(Unitary rel. from M_a to $\tilde{M}_a$).

(6) $\mathcal{U} = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X$

$\Rightarrow \mathcal{U} : |\psi\rangle|0\rangle_E \longrightarrow \sum_a M_a|\psi\rangle|a\rangle_E$

<u>Here:</u> $M_0 = \frac{X}{\sqrt{2}}$, $M_1 = \frac{Y}{\sqrt{2}}$.

Thus, quantum channel in operator-sum (= Kraus) representation reads

$$\mathcal{E}(\rho) = \sum_{a=0}^{1} M_a \rho M_a^\dagger =$$

$$= \frac{1}{2}\left[X\rho X + Y\rho Y\right].$$

(5)

Gate fidelity

$$F(\mathcal{U}, \mathcal{E}) = \min_{|\psi\rangle} F\Big(\mathcal{U}|\psi\rangle\langle\psi|,\ \mathcal{E}(|\psi\rangle\langle\psi|)\Big)$$

(a) $\mathcal{U} = X$

$\tilde{Z} = XY$

$$\mathcal{E} = (1-p)\, X\rho X + p\, \tilde{Z}\rho\tilde{Z}$$

Fidelity for given $|\psi\rangle = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$.

$$F = \mathrm{Tr}\sqrt{\underbrace{\sqrt{X|\psi\rangle\langle\psi|X}}_{\text{still pure state}}\ \mathcal{E}(|\psi\rangle\langle\psi|)\sqrt{X|\psi\rangle\langle\psi|X}} =$$

$$= \mathrm{Tr}\sqrt{\underbrace{X|\psi\rangle\langle\psi|X}_{=X\rho X}\big[(1-p)X\rho X + p\, Z\rho Z\big]\, X\rho X} =$$

$$= \mathrm{Tr}\sqrt{(1-p)\, X|\psi\rangle\langle\psi|X + p\, X|\psi\rangle\underbrace{\langle\psi|-iY|\psi\rangle}_{=-i r_y}\underbrace{\langle\psi|iY|\rho X}_{=i r_y\langle\psi|}}$$

$$= \mathrm{Tr}\sqrt{(1-p)\, X|\psi\rangle\langle\psi|X + p\, r_y^2\, X|\psi\rangle\langle\psi|X}$$

$$= \mathrm{Tr}\sqrt{(1-p+p r_y^2)\, \underbrace{X|\psi\rangle\langle\psi|X}_{=|\psi'\rangle\langle\psi'|}} =$$

$$= \sqrt{1-p+p r_y^2} \quad , \; r_y = \sin\theta\cos\phi$$

Now minimize over all $|\psi\rangle$:

$$F(\mathcal{U}, \mathcal{E}) = \min_{|\psi\rangle} \sqrt{1-p(1-r_y^2)} =$$

$$= \min_{\substack{r_y \\ r_y \in [-1,1]}} \sqrt{1-p(1-r_y^2)} = \boxed{\sqrt{1-p}}\,.$$

We know that the fidelity between a pure and a mixed state reads

$$F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle}$$

Here: $|\psi\rangle = X|\psi_0\rangle$

$\rho = \mathcal{E}(|\psi_0\rangle\langle\psi_0|)$.

$$\Rightarrow \quad F\left(X|\psi_0\rangle ,\ \mathcal{E}(\rho) \right) =$$

$$= \sqrt{\left(\langle\psi_0| X\right) \mathcal{E}(\rho) \left(X|\psi_0\rangle\right)} =$$

$$= \sqrt{\langle\psi_0| X \left[(1-p) X|\psi_0\rangle\langle\psi_0| X + p\, Z|\psi_0\rangle\langle\psi_0| Z \right] \cdot X|\psi_0\rangle}$$

$$= \sqrt{(1-p) + p \langle\psi_0| \underbrace{XZ}_{=-iY} |\psi_0\rangle\langle\psi_0| \underbrace{ZX}_{=iY} |\psi_0\rangle} =$$

$$= \sqrt{1-p+p\, r_y^2} \, .$$

(b)

$\mathcal{E}$ is a quantum channel that approximates $\mathcal{U}$

$\mathcal{F}$ — " — $\mathcal{V}$

Define the error

$$E(\mathcal{U}, \mathcal{E}) = \max_{\rho} d(\mathcal{U}\rho\mathcal{U}^{\dagger}, \mathcal{E}(\rho)),$$

any distance measure, e.g.

$d = a\cos[F(\mathcal{U}\rho\mathcal{U}^{\dagger}, \mathcal{E}(\rho))]$.

Show:

$$E(V\mathcal{U}, \mathcal{F}\circ\mathcal{E}) \leq E(\mathcal{U}, \mathcal{E}) + E(V, \mathcal{F}).$$

Note, the metric fulfills the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

Using Uhlmann's theorem, one can show that $a\cos(F(\rho, \sigma))$ fulfills this

$$d(\rho, \tau) \leq \underbrace{d(\rho, \sigma)}_{= d(\sigma, \rho)} + d(\sigma, \tau)$$ (Nielsen Eq. (9.86))

According to Nielsen exercise (9.18), it holds that

$$d(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq d(\rho, \sigma) \quad [\text{for } d = \mathrm{acos}(\cdot,\cdot)]$$

for any TP map.

We also know that d is invariant under unitary tf as

$$F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)$$

Thus,

$= \mathrm{acos}(F(\rho,\sigma)) = d_F = d$

$$E(UV, \mathcal{F}\circ\mathcal{E}) = \max_{\rho} d_F(UV\rho V^\dagger U^\dagger, (\mathcal{F}\circ\mathcal{E})(\rho))$$

It holds $F(UV\rho V^\dagger U^\dagger, (\mathcal{F}\circ\mathcal{E})(\rho)) = F[\rho, V^\dagger U^\dagger(\mathcal{F}\circ\mathcal{E})(\rho)\, UV]$.

Define $\widetilde{UV}$ as $UV\rho(UV)^\dagger$, then

$$\Rightarrow F(\widetilde{UV}\rho, (\mathcal{F}\circ\mathcal{E})(\rho)) = F[\rho, \widetilde{V^\dagger U^\dagger(\mathcal{F}\circ\mathcal{E})(\rho)}]$$

First use the triangle inequality

$$d\left[(\mathcal{F}\circ\mathcal{E})(\rho),\ \overline{\mathcal{V}\mathcal{U}}\,\rho\right] \leq d\left[(\mathcal{F}\circ\mathcal{E})(\rho),\ \mathcal{F}\circ\overline{\mathcal{U}}\rho\right] + d\left[\mathcal{F}\circ\overline{\mathcal{U}}\rho,\ \overline{\mathcal{V}\mathcal{U}}\,\rho\right]$$

The first term on the RHS obeys (using contraction property)

$$d\Big[\mathcal{F}\circ\underbrace{(\mathcal{E}(\rho))}_{\hat{=}\rho'},\ \mathcal{F}\circ\underbrace{\overline{\mathcal{U}}\rho}_{\hat{=}\sigma'}\Big] \leq d\Big[\underbrace{\mathcal{E}(\rho)}_{=\rho'},\ \underbrace{\overline{\mathcal{U}}\rho}_{=\sigma'}\Big]$$

We now use the definition of the error:

$$E\left(\mathcal{E}(\rho),\ \overline{\mathcal{U}}\rho\right) = \max_{\rho} d\left[\mathcal{E}(\rho),\ \overline{\mathcal{U}}\rho\right]$$

When considering the $\max_{\rho}$ operation, the second term on the RHS obeys

$$\max_{\rho} d\left[\mathcal{F}\circ\overline{\mathcal{U}}\rho,\ \overline{\mathcal{V}\mathcal{U}}\,\rho\right] = \max_{\rho'} d\left[\mathcal{F}(\rho'),\ \overline{\mathcal{V}}\rho'\right]$$

Therefore,

$$E\left[\mathcal{F}\circ\mathcal{E},\ \overline{\mathcal{V}\mathcal{U}}\right] = \max_{\rho} d\left[(\mathcal{F}\circ\mathcal{E})(\rho),\ \overline{\mathcal{V}\mathcal{U}}\,\rho\right]$$

$$\leq \max_{\rho} \left\{ d\left[(\mathcal{F}\circ\mathcal{E})(\rho), \mathcal{F}\widetilde{\mathcal{U}}\rho\right] + d\left[\mathcal{F}\widetilde{\mathcal{U}}\rho, \widetilde{\mathcal{V}\mathcal{U}}\rho\right]\right\}$$

$$\leq \max_{\rho} \left\{ d\left[\mathcal{E}(\rho), \widetilde{\mathcal{U}}\rho\right] + d\left[\mathcal{F}\rho, \widetilde{\mathcal{V}}\rho\right]\right\}$$

$$= E(\mathcal{E}, \widetilde{\mathcal{U}}) + E(\mathcal{F}, \widetilde{\mathcal{V}}) \qquad \square$$