(1)

$$\mathcal{E}(\rho) = |+\rangle\langle+|\rho|+\rangle\langle+| + |-\rangle\langle-|\rho|-\rangle\langle-| =$$

$$= P_+ \rho P_+ + P_- \rho P_-$$ (Kraus description of quantum channel)

with $P_+ = |+\rangle\langle+|$, $P_- = |-\rangle\langle-|$

and $|\pm\rangle = \frac{1}{\sqrt{2}}[|0\rangle \pm |1\rangle]$

In terms of Pauli operators, we find

$$P_+ = \frac{1}{2}[|0\rangle + |1\rangle][\langle 0| + \langle 1|] = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}(I + X)$$

$$P_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}(I - X).$$

Note: $P_+^\dagger P_+ + P_-^\dagger P_- = P_+ + P_- = I$ as required for Kraus operators.

$\Rightarrow$ Pauli basis : $P_\mu \in \left\{ \frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Y}{\sqrt{2}}, \frac{Z}{\sqrt{2}} \right\}$

$$\Rightarrow P_\pm = \frac{1}{\sqrt{2}}[P_0 \pm P_1]$$

$\Rightarrow \quad \mathcal{E}(\rho) = P_+ \rho P_+ + P_- \rho P_- =$

$$= \frac{1}{2}(P_0 + P_1)\rho(P_0 + P_1)$$

$$+ \frac{1}{2}(P_0 - P_1)\rho(P_0 - P_1) =$$

$$= P_0 \rho P_0 + P_1 \rho P_1 = \sum_{a,b} \chi_{ab} P_a \rho P_b$$

$\Rightarrow$ Process χ-matrix in Pauli basis reads

$$\boxed{(\chi_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}.$$

CP iff eigenvalues of $\chi \geq 0 \Rightarrow$ this is the case

$\Rightarrow \mathcal{E}$ is CP.

(b) PTM (R_ε) is obtained from χ_{ab} as

$$(R_\varepsilon)_{ij} = \mathrm{Tr}\left[\chi\, P_j^T \otimes P_i\right]$$

Or directly

$$(R_\varepsilon)_{ij} = \mathrm{Tr}\left[P_i\, \mathcal{E}(P_j)\right] = \frac{1}{2}\mathrm{Tr}\left[P_i\, \mathcal{E}(P_j)\right]$$

Here:

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1 \quad \text{(bit flip channel)}$$

$$\Rightarrow (R_\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}(P_0) = \frac{P_0}{2} + \frac{P_0}{2} = P_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \Rightarrow \mathrm{Tr}\left[P_i \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right] =$$

$$= \delta_{i,0}$$

$$\mathcal{E}(P_1) = \frac{P_1}{2} + \frac{P_1}{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}\left[P_i \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \delta_{i,1}$$

$$\mathcal{E}(P_2) = P_2 - P_2 = 0, \quad \mathcal{E}(P_3) = 0.$$

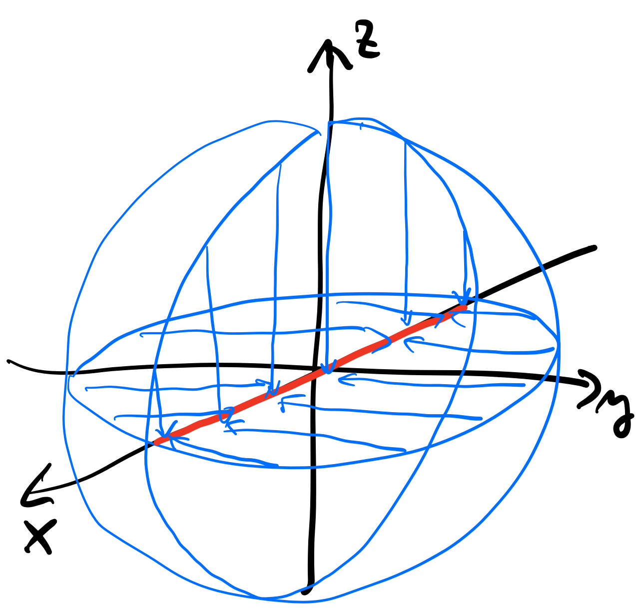
Effect on state $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$

$$\Rightarrow |\rho\rangle\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix}$$

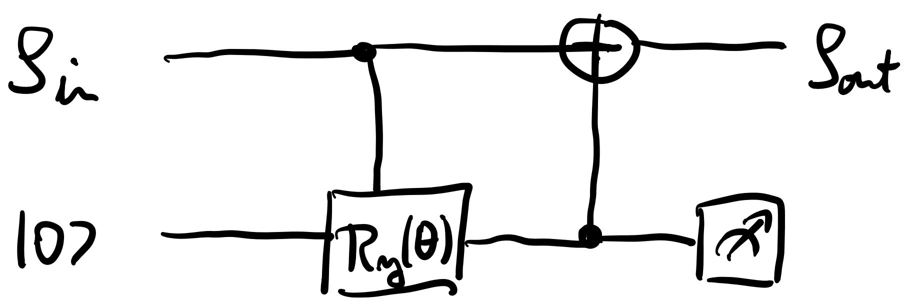
$$\Rightarrow |\rho'\rangle\rangle = \left(R_{\mathcal{E}}\right)|\rho\rangle\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ 0 \\ 0 \end{pmatrix}$$

$\hat{=}$ $\rho' = \frac{1}{2}(I + r_x X) \Rightarrow$ projection onto X axis.

$$\vec{r}\,' = (r_x, 0, 0)$$



(2) (a) Circuit model for amplitude damping



$\rho_{in} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = a|0\rangle + b|1\rangle$, $a = \cos\frac{\alpha}{2}$, $b = \sin\frac{\alpha}{2} e^{i\beta}$

$\Rightarrow$ $|0\rangle|\psi\rangle \xmapsto{CR_y} a|0\rangle|0\rangle + b\big(\underbrace{R_y(\theta)|0\rangle}\big)|1\rangle =$

$= e^{-i\frac{\theta Y}{2}}|0\rangle = \left(\cos\frac{\theta}{2} - iY\sin\frac{\theta}{2}\right)|0\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle - i\sin\frac{\theta}{2}\, i|1\rangle.$

$= a|0\rangle|0\rangle + b\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\right)|1\rangle$

$\Big\downarrow$ CNOT

$a|0\rangle|0\rangle + b\cos\frac{\theta}{2}|0\rangle|1\rangle + b\sin\frac{\theta}{2}|1\rangle|0\rangle =$

$= |0\rangle\Big(\underbrace{a|0\rangle + b\cos\frac{\theta}{2}|1\rangle}_{= E_0|\psi\rangle}\Big) + |1\rangle \underbrace{b\sin\frac{\theta}{2}|0\rangle}_{= E_1|\psi\rangle}$

Now we measure the ancilla in the computational basis.

We find outcome $|1\rangle$ with probability $p_1 = |b|^2 \sin^2\frac{\theta}{2}$ and outcome $|0\rangle$ with probability

$$p_0 = \underbrace{|a|^2}_{=1-|b|^2} + |b|^2\cos^2\frac{\theta}{2} \underset{|a|^2+|b|^2=1}{=} 1 - |b|^2\overbrace{\left(1-\cos^2\frac{\theta}{2}\right)}^{=\sin^2\frac{\theta}{2}} = 1 - p_1 .$$

The output state is thus

$$\left(a|0\rangle + b\cos\frac{\theta}{2}|1\rangle\right)\left(a\langle 0| + b\cos\frac{\theta}{2}\langle 1|\right)$$

$$+ |b|^2 \sin^2\frac{\theta}{2}\, |0\rangle\langle 0|$$

The Kraus operators can be calculated from the transition "amplitudes":

$$M_r = \langle r| U_{\text{circuit}} |0\rangle$$

or (N&C 8.37): $\quad U|\psi\rangle|0\rangle = \sum_r M_r |\psi\rangle|r\rangle .$

$$\Rightarrow M_0 \underbrace{|\psi\rangle}_{} = a|0\rangle + b\cos\frac{\theta}{2}|1\rangle = \begin{pmatrix} a \\ b\cos\frac{\theta}{2} \end{pmatrix}$$

$$= a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix}$$

$$M_1|\psi\rangle = b\sin\frac{\theta}{2}|0\rangle = \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ 0 & 0 \end{pmatrix}$$

$$M_1\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b\sin\frac{\theta}{2} \\ 0 \end{pmatrix}$$

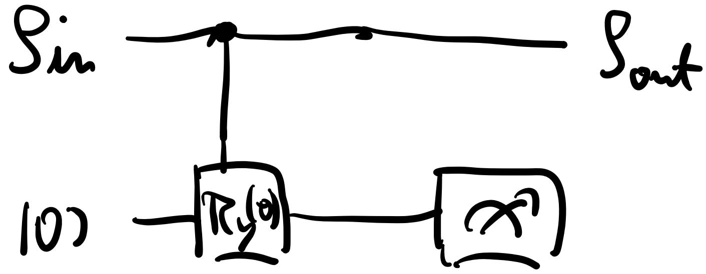
Amplitude damping Kraus operators are of the form

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $\boxed{\text{Circuit realizes amplitude damping with parameter } \gamma = \sin^2\frac{\theta}{2}.}$

(b)

Circuit for phase damping



Similar to (a) (see above):

$$|0\rangle|\psi\rangle \longrightarrow a|0\rangle|0\rangle + b\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\right)|1\rangle =$$

$$= |0\rangle\left(a|0\rangle + b\cos\frac{\theta}{2}|1\rangle\right) + |1\rangle\, b\sin\frac{\theta}{2}|1\rangle$$

$$\Rightarrow M_0|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix}$$

$$M_1|\psi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & \sin\frac{\theta}{2} \end{pmatrix}$$

The Kraus operators of the phase damping channel read

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

$\Rightarrow$ identify $\lambda = \sin^2 \frac{\theta}{2}$ for circuit to model the phase damping channel.

③

ⓐ

Depolarizing channel

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{I}{d} \quad , d=2 \text{ for single qubit}$$

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}), \quad \sigma = \frac{1}{2}(I + \vec{s}\cdot\vec{\sigma})$$

$$\Rightarrow \mathcal{E}(\rho) = \frac{1}{2}\left[I + (1-p)\vec{r}\cdot\vec{\sigma}\right]$$

$$\mathcal{E}(\sigma) = \text{accordingly}$$

$\Rightarrow$ trace distance $D(\rho,\sigma)$: we showed that

$$D(\rho,\sigma) = \frac{1}{2}|\vec{r}-\vec{s}|.$$

Therefore:

$$D[\mathcal{E}(\rho),\mathcal{E}(\sigma)] = \frac{1}{2}(1-p)|\vec{r}-\vec{s}|$$

Thus $D[\mathcal{E}(\rho),\mathcal{E}(\sigma)] = (1-p)\,\mathcal{E}(\rho,\sigma) < \mathcal{E}(\rho,\sigma)$

for $p>0$. Strictly contractive.

(6) Bit flip channel

Kraus operators $M_0 = \sqrt{1-p}\, I$, $M_1 = \sqrt{p}\, X$.

Assume $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \Rightarrow |\rho\rangle\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix}$

$$\Rightarrow \mathcal{E}(\rho) = (1-p)\rho + p\, X\rho X =$$

$$= (1-p)\frac{I + \vec{r}\cdot\vec{\sigma}}{2} + p\left[\frac{I}{2} + \frac{r_x X}{2} - \frac{r_y Y}{2} - \frac{r_z Z}{2}\right]$$

$$= \frac{1}{2}\left[I + r_x X + (1-2p)(r_y Y + r_z Z)\right]$$

$$\Rightarrow |\mathcal{E}(\rho)\rangle\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ r_x \\ (1-2p)\, r_y \\ (1-2p)\, r_z \end{pmatrix}$$

$$\Rightarrow D[\mathcal{E}(\rho), \mathcal{E}(\sigma)] = \frac{1}{2}\left|\begin{pmatrix} 0 \\ r_x - s_x \\ (1-2p)(r_y - s_y) \\ (1-2p)(r_z - s_z) \end{pmatrix}\right| =$$

$$= \frac{1}{2}\sqrt{(r_x - s_x)^2 + (1-2p)^2\left[(r_y - s_y)^2 + (r_z - s_z)^2\right]}$$

$$\leq \frac{1}{2}\sqrt{(r_x - s_x)^2 + (r_y - s_y)^2 + (r_z - s_z)^2} = D(\rho, \sigma).$$

Equality holds if $r_y = s_y$ and $r_z = s_z$.

Fixed points:

$$\mathcal{E}(\rho) = \rho \Rightarrow \{(r_x, r_y, r_z) ; \; r_y = r_z = 0\}$$

(points on the x-axis of the Bloch ball are fixed points of the bit flip channel).