**SAARLAND UNIVERSITY**
**Faculty of Mathematics and Computer Science**
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**Assignment for the lecture Functional Analysis**
Winter term 2022/23

**Sheet 10**
Due on Mon 23.1.2023

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**Exercise 1** (10 Points)**.** Consider the Banach algebra $l_1(\mathbb{Z})$, endowed with the convolution $*$ as multiplication operation (c.f. Sheet 9, Ex. 1). Show that every character of $l_1(\mathbb{Z})$ (i.e. complex homomorphism $l_1(\mathbb{Z}) \to \mathbb{C}$) is of the form

$$\varphi_z \colon l_1(\mathbb{Z}) \to \mathbb{C}, \quad \varphi_z((a_n)_{n\in\mathbb{Z}}) := \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\in\mathbb{Z}} a_n e^{int}$$

for some $z = e^{it} \in \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$.

**Exercise 2** (10 Points)**.** Let $(A, \|\cdot\|_A)$ be a Banach algebra and $J$ a closed ideal in $A$. We endow the quotient vector space

$$A/J = \{a + J \colon a \in A\} \qquad \text{[i.e., } a + J = b + J \text{ iff } a - b \in J]$$

with

$$\begin{aligned} \lambda(a+J) &:= \lambda a + J \\ (a+J) + (b+J) &:= (a+b) + J \\ (a+J)(b+J) &:= ab + J \\ \|a+J\| &:= \inf_{x\in J} \|a+x\|_A = \operatorname{dist}(a, J) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $a, b \in A$. Show that $\|\cdot\|$ defines a norm on $A/J$ and that $A/J$ is complete w.r.t. to this norm. Conclude that $(A/J, \|\cdot\|)$ is a Banach algebra.

**Exercise 3** (10 Points)**.** Consider $M_n(\mathbb{C})$, the $n \times n$ matrices over $\mathbb{C}$. Show that, for $n \geq 2$, there does not exist any complex homomorphism $M_n(\mathbb{C}) \to \mathbb{C}$.

**Exercise 4** (10 Points)**.** Let $V$ be the Volterra integration operator on $L^2(0,1)$, given by

$$(Vf)(x) = \int_0^x f(y)dy \qquad (f \in L^2(0,1)).$$

Prove that

$$\|V\| = \frac{2}{\pi}.$$

[Hint: Express $V^*V$ as an integral operator. By differentiation convert the equation $V^*Vf = \lambda f$ into a differential equation and solve it.]