SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Prof. Dr. Roland Speicher
M.Sc. Felix Leid

Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 11
Due on Mon 30.1.2023

Exercise 1 (10 Points)**.** Let $a \in A$ be an element in a Banach algebra A and let

$$p(z) = \sum_{k=0}^{n} \alpha_k z^k \qquad (\text{with } \alpha_0, \dots, \alpha_n \in \mathbb{C})$$

be a polynomial in the variable z. Then we can apply the polynomial to our operator a,

$$p(a) := \sum_{k=0}^{n} \alpha_k a^k \in A, \qquad \text{where} \quad a^0 = 1 \in A.$$

Show the spectral mapping theorem for this situation, i.e.,

$$\sigma(p(a)) = p(\sigma(a)) := \{p(\lambda) \mid \lambda \in \sigma(a)\}$$

Exercise 2 (10 Points)**.** Let x be an idempotent in a Banach algebra, i.e., $x^2 = x$.

(i) By providing explicit formulas for the inverse of $\lambda - x$, for $\lambda \neq 0, 1$, show that the spectrum of x can only consist of 0 or 1.

(ii) Show that $\sigma(x) = \{0, 1\}$ unless $x = 0$ or $x = 1$.

(iii) Show by an example that we can have $\|x\| > 1$ for an idempotent x.

Exercise 3 (10 Points)**.** Let x and y be two commuting idempotent elements in a Banach algebra; i.e., $x^2 = x$, $y^2 = y$, $xy = yx$. Show that either $x = y$ or $\|x - y\| \geq 1$. Is this still true if x and y don't commute?
[Hint: For the first part consider $[1 - (x - y)][-1 - (x - y)](x - y)$.]

Exercise 4 (10 Points)**.** Let $u \in A$ be a unitary element in a C^*-algebra A, i.e., $u^*u = 1 = uu^*$. Show that the spectrum of u satisfies:

$$\sigma(u) \subset \mathbf{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$