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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 2
Due on Mon 14.11.2022, hand in before the lecture.

Exercise 1 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space and $A \subseteq \mathcal{H}$ be an arbitrary subset. Prove the following assertions:

a) $A^\perp$ is a closed linear subspace of $\mathcal{H}$.

b) $(A^\perp)^\perp = \overline{\text{span A}}$, the closure of the span of A.

Exercise 2 (10 Points)**.** Consider the Hilbert space $L^2(-1,1)$ endowed with the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(t)\overline{g(t)}\mathrm{d}t.$$

Apply the Gram-Schmidt Orthogonalization Procedure to the sequence of functions $\{f_n\}_{n=0}^\infty$ given by

$$f_n : [-1,1] \to \mathbb{C}, \quad t \mapsto t^n$$

and show that the resulting ONB $(e_n)_{n=0}^\infty$ is of the form

$$e_n(t) = \frac{1}{\sqrt{2}}(2n+1)^{\frac{1}{2}} P_n(t)$$

where

$$P_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n,$$

is the n-th Legendre Polynomial.

$*$ If you are not familiar with $L^2(-1,1)$, you may consider the pre-Hilbert space $C[-1,1]$ with the same inner product instead.

Exercise 3 (10 Points)**.** Consider the Hilbert space $(l_2, \langle , \rangle)$ of complex square-summable sequences given by

$$l_2 := \left\{ (a_n)_{n=1}^{\infty} \colon a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n| < \infty \right\}, \quad \langle (a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \rangle := \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

For which $\lambda \in \mathbb{C}$ is

$$L \colon l_2 \to \mathbb{C}, \quad L((a_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \lambda^n a_n$$

well defined? Show that L defines a bounded (i.e., continuous) linear functional for these λ and compute its norm.

Exercise 4 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space, $(x_n)_{n=1}^{\infty}$ a sequence of elements of $\mathcal{H}$ and $x \in \mathcal{H}$.

- We say $(x_n)_{n=1}^{\infty}$ converges (in norm) to x if

$$\|x_n - x\| \to 0 \text{ for } n \to \infty.$$

- We say $(x_n)_{n=1}^{\infty}$ converges weakly to x, if for any $y \in \mathcal{H}$ we have

$$\langle x_n, y \rangle \to \langle x, y \rangle \text{ for } n \to \infty.$$

Show the following.

a) If $(x_n)_{n=1}^{\infty}$ converges in norm to x then it also converges weakly to x.
(Norm convergence implies weak convergence)

b) If $(x_n)_{n=1}^{\infty}$ converges weakly to x and weakly to x', then $x = x'$.
(Uniqueness of weak limits)

c) Let $(e_n)_{n=1}^{\infty}$ be an orthonormal system in $\mathcal{H}$. Then $(e_n)_{n=1}^{\infty}$ converges weakly to zero but not in norm.
(Weak convergence does not imply norm convergence)