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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 3
Due on Mon 21.11.2022, hand in before the lecture.

Exercise 1 (10 Points)**.** For $n \in \mathbb{N}$ we denote by $M_n(\mathbb{C})$ the algebra of $n \times n$ matrices with complex entries, and for a matrix $A \in M_n(\mathbb{C})$ we denote the set of its eigenvalues by $\sigma(A)$.

a) Show that the operator norm of the linear map $\mathbb{C}^n \to \mathbb{C}^n$, defined by a matrix $A \in M_n(\mathbb{C})$ is finite. (This justifies the isomorphism $B(\mathbb{C}^n) \cong M_n(\mathbb{C})$)

b) Let $A \in M_n(\mathbb{C})$ normal, i.e. $A^*A = AA^*$. Show that

$$\|A\| = \max_{\lambda \in \sigma(A)} |\lambda|.$$

[The following fact from linear algebra might be useful for this: a matrix A is normal if and only if it can be diagonalized with the help of unitary matrix; i.e., there is a unitary matrix U such that U^*AU is diagonal.]

c) Does b) hold if A is not normal? Justify your answer.

Exercise 2 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. We define the *matrix coefficients* $(a_{n,m})_{n,m=1}^\infty$ of an operator $A \in B(\mathcal{H})$ by

$$a_{n,m} = \langle Ae_m, e_n \rangle \quad \text{for all } n, m \in \mathbb{N}.$$

a) Show that $A \in B(\mathcal{H})$ is uniquely determined by its matrix coefficients.

b) We put $C = \|A\|$. Show that

$$\sup_{m \in \mathbb{N}} \sum_{n=1}^\infty |a_{n,m}|^2 \leq C^2 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \sum_{m=1}^\infty |a_{n,m}|^2 \leq C^2$$

c) Now consider the Hilbert space $\mathcal{H} = l_2$. Construct a family $(a_{n,m})_{n,m=1}^\infty$ of complex numbers such that the inequalities of b) hold for some finite $C \geq 0$, but $(a_{n,m})_{n,m=1}^\infty$ are not the matrix coefficients of an operator $A \in B(l_2)$.

Exercise 3 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_0 \subseteq \mathcal{H}$ a closed linear subspace. We call the operator $P\colon \mathcal{H} \to \mathcal{H}$ with

$$\|x - Px\| = \operatorname{dist}(x, \mathcal{H}_0) \quad \text{for all } x \in \mathcal{H}$$

the *orthogonal projection* onto $\mathcal{H}_0$; compare Exercise 4 of Assignment 1.
Show that a map $P\colon \mathcal{H} \to \mathcal{H}$ is an orthogonal projection onto a closed linear subspace if and only if $P \in B(\mathcal{H})$ and $P^2 = P = P^*$.

Exercise 4 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space and $V \in B(\mathcal{H})$. We call V a *partial isometry* if we have $VV^*V = V$. Show that the following are equivalent.

i) V is a partial isometry.

ii) V^*V is an orthogonal projection.

iii) VV^* is an orthogonal projection.

iv) There is a closed linear subspace $\mathcal{K} \subseteq \mathcal{H}$, such that $V|_{\mathcal{K}}\colon \mathcal{K} \to \mathcal{H}$ is an isometry (i.e., it satisfies $\|Vx\| = \|x\|$ for all $x \in \mathcal{K}$) and $V|_{\mathcal{K}^\perp} \equiv 0$.

How are the images of the orthogonal projections V^*V and VV^* from ii) and iii) related to the closed linear subspace $\mathcal{K} \subseteq \mathcal{H}$ from iv)?