SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 4
Due on Mon 28.11.2022, hand in before the lecture.

Exercise 1 (10 Points). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($a_n \in \mathbb{C}$) be a power series with radius of convergence $0 < R \leq \infty$.

a) Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$ such that $\|A\| < R$. Show that there is a well defined operator $f(A) \in B(\mathcal{H})$.

b) Now consider $f(z) = e^z$. Show that e^{iA} is unitary if A is self-adjoint.

Exercise 2 (10 Points). a) Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Show that the image $\mathrm{ran}(A)$ of any operator $A \in B(\mathcal{H})$ is a closed set.

b) Construct an example of a bounded operator $A \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, such that $\mathrm{ran}(A)$ is not closed.

Exercise 3 (20 Points). Let $1 \leq p \leq \infty$ and consider the Banach spaces

$$l_p := \left\{ (a_n)_{n=1}^{\infty} \mid a_n \in \mathbb{C}, \|(a_n)_{n=1}^{\infty}\|_p < \infty \right\},$$

where

$$\|(a_n)_{n=1}^{\infty}\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \quad (1 \leq p < \infty) \qquad \text{and} \qquad \|(a_n)_{n=1}^{\infty}\|_{\infty} := \sup_{n \in \mathbb{N}} |a_n|.$$

a) Let $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{q} + \frac{1}{p} = 1$. Show that the dual space of l_p is isometrically isomorphic to l_q.

Hint. Use Hölders inequality $\|(a_n b_n)_{n=1}^{\infty})\|_1 \leq \|(a_n)_{n=1}^{\infty}\|_q \cdot \|(b_n)_{n=1}^{\infty}\|_p$.

b) Let $1 \leq p < q \leq \infty$. Show that $l_p \subseteq l_q$, $l_p \neq l_q$ and $\|(a_n)_{n=1}^{\infty}\|_q \leq \|(a_n)_{n=1}^{\infty}\|_p$.

c) Let $1 \leq p < \infty$ and $(a_n)_{n=1}^{\infty} \in l_p$ (and thus also in all l_q with $q \geq p$). Show that

$$\lim_{q \to \infty} \|(a_n)_{n=1}^{\infty}\|_q = \|(a_n)_{n=1}^{\infty}\|_{\infty}.$$

d) We can define l_p as above also for $0 < p < 1$. Why is this not a normed space?