SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 5
Due on Mon 05.12.2022, hand in before the lecture.

Exercise 1 (10 Points)**.** We consider the space $(c, \|\cdot\|_\infty)$ of convergent complex sequences, endowed with the sup norm, i.e.

$$c = \left\{(a_n)_{n=1}^\infty : a_n \in \mathbb{C}, \lim_{n\to\infty} a_n \text{ exists}\right\}, \quad \|(a_n)_{n=1}^\infty\|_\infty = \sup_{n\in\mathbb{N}} |a_n|.$$

a) Show that the function F that maps a sequence to its limit, i.e.

$$F : c \to \mathbb{C}, \quad (a_n)_{n=1}^\infty \mapsto \lim_{n\to\infty} a_n$$

is linear and continuous.

b) Show that the space

$$c_0 = \{(a_n)_{n=1}^\infty : a_n \in \mathbb{C}, \lim_{n\to\infty} a_n = 0\}$$

of complex sequences converging to 0 is a closed linear subspace of c.

c) Determine all continuous linear functionals on c.

Exercise 2 (10 Points)**.** Show that there is a continuous function $g \in C[0,1]$, which is nowhere differentiable in $[0, \frac{1}{2}]$. Proceed as follows:

a) Show that the sets

$$F_n := \left\{ f \in C[0,1] : \exists x \in [0, 1/2] \text{ with } \sup_{0<h<\frac{1}{2}} \frac{|f(x+h) - f(h)|}{h} \leq n \right\}$$

are closed and have no inner points, for all $n \in \mathbb{N}$.

b) Conclude by using Baire's theorem.

Exercise 3 (10 Points)**.** Let $l_\infty^{\mathbb{R}}$ denote the Banach space of real-valued bounded sequences endowed with the sup norm $\|\cdot\|_\infty$. Show that

$$q\colon l_\infty^{\mathbb{R}} \to \mathbb{R}, \quad (a_n)_{n=1}^\infty \mapsto \limsup_{n\to\infty} a_n$$

defines a sublinear functional. Conclude that there is a linear bounded functional $L\colon l_\infty^{\mathbb{R}} \to \mathbb{R}$ with

$$\liminf_{n\to\infty} a_n \leq L((a_n)_{n=1}^\infty) \leq \limsup_{n\to\infty} a_n.$$

This means, in a way we are able to define *limits* of bounded (not necessarily converging) sequences. (In particular one obtains the limit if we restrict L to converging sequences.)

Exercise 4 (10 Points)**.** Show that there is no real or complex Banach space of countably infinite vector space dimension.
[Dimension is here in the sense of linear algebra, i.e., the cardinality of a basis; and a basis allows to write any element in a unique way as a finite linear combination. Recall also that any vector space has, by the axiom of choice, a basis and that all bases have the same cardinality.]