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Faculty of Mathematics and Computer Science
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 6
Due on Mon 12.12.2022, hand in before the lecture.

Exercise 1 (10 Points)**.** A sequence $(a_n)_{n=1}^\infty$ of complex numbers is called finite if a_n is different from zero for only finitely many $n \in \mathbb{N}$. We consider in l_∞ the subspace of finite sequences

$$V := \{(a_n)_{n=1}^\infty \colon a_n \neq 0 \text{ for only finitely many } n \in \mathbb{N}\} \subset l_\infty$$

and define a linear map on V by

$$A \colon V \to V, \quad (a_n)_{n=1}^\infty \mapsto (na_n)_{n=1}^\infty,$$

i.e. A multiplies a sequence with the sequence $(n)_{n=1}^\infty$ element-wise.

a) Show that V is not closed in l_∞ (and thus not complete).

b) Show that A is not continuous on $(V, \|\cdot\|_\infty)$ (we endow V with the sup-norm of l_∞).

c) Show that A is the pointwise limit of continuous linear mappings $A_n \colon V \to V$.

Remark. This exercise shows that we cannot drop the assumption of completeness in the theorem of Banach-Steinhaus.

Exercise 2 (10 Points)**.** Let $k \in C([0,1] \times [0,1])$ be a continuous function. We define the *integral operator K with kernel k* by

$$K \colon C([0,1]) \to C([0,1]), \quad (Kf)(s) := \int_0^1 k(s,t) f(t) \mathrm{d}t.$$

Show that K is a compact operator on $C([0,1])$.
Hint: You can use the following theorem that characterises the compact sets of $C([0,1])$.

Theorem (Arzela-Ascoli Theorem)*.* A subset $A \subseteq C([0,1])$ is compact if and only if it is closed, bounded and *equicontinuous*, i.e. it holds that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in A \quad \forall x, y \in [0,1] \text{ with } |x - y| < \delta \colon \quad |f(x) - f(y)| < \varepsilon.$$

Exercise 3 (10 Points)**.** Let X, Y be Banach spaces. The product space

$$X \times Y = \{(x, y) \colon x \in X, y \in Y\}$$

equipped with component-wise addition and scalar multiplication is a vector space. We define on it a norm by

$$\|(x, y)\| := \|x\|_X + \|y\|_Y$$

and denote it then by $X \oplus_1 Y$; and its elements by $x \oplus y := (x, y)$.

a) Show that $X \oplus_1 Y$ is a Banach space.

b) Let $A \colon X \to Y$ be a linear map and

$$\operatorname{graph}(A) := \{x \oplus Ax \colon x \in X\} \subseteq X \oplus_1 Y$$

be the *graph* of A. Show that $\operatorname{graph}(A)$ is a closed subset of $X \oplus_1 Y$ if A is continuous.

c) Prove the *Closed Graph Theorem*: If $\operatorname{graph}(A)$ is a closed subset of $X \oplus_1 Y$ then A is continuous.

Exercise 4 (10 Points)**.** Let X be a Banach space and $\mathcal{K}(X) \subseteq B(X)$ be the set of compact operators on X. Show that $\mathcal{K}(X)$ is a closed linear subspace of $B(X)$ and a two-sided ideal in $B(X)$.