**SAARLAND UNIVERSITY**
**Faculty of Mathematics and Computer Science**
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**Assignment for the lecture Functional Analysis**
Winter term 2022/23

**Sheet 7**
Due on Mon 19.12.2022

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**Exercise 1** (10 Points)**.** Let $S\colon l_2 \to l_2$ be the one-sided shift, which is determined by

$$S(a_1, a_2, a_3 \dots) = (0, a_1, a_2, a_3 \dots).$$

a) Calculate $S^*$.

b) Show that $\sigma_p(S) = \emptyset$.

c) Determine the point spectrum of $S^*$.

**Exercise 2** (10 Points)**.** Let $A \in B(\mathcal{H})$. We say that $\lambda \in \mathbb{C}$ has a sequence of approximating eigenvectors if there is a sequence $(x_n)_{n\in\mathbb{N}}$ of vectors $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ for all $n \in \mathbf{N}$ such that

$$\|Ax_n - \lambda x_n\| \to 0.$$

Show that any $\lambda \in \mathbb{C}$ which admits such a sequence of eigenvectors belongs to the spectrum of $A$.

**Exercise 3** (10 Points)**.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with ONB $(e_n)_{n=1}^\infty$ and let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers such that

$$M := \sup_{n\in\mathbb{N}} |a_n| < \infty.$$

a) Show that the prescription $Ae_n = a_n e_n$ $(n \in \mathbb{N})$ defines uniquely a bounded operator $A$ on $\mathcal{H}$ and that we have $\|A\| = M$.
(Such an operator is called diagonalizable.)

b) Prove that $A$ is compact if and only if $\lim_{n\to\infty} a_n = 0$.

c) Calculate the point spectrum $\sigma_p(A)$.

d) Speculate about the spectrum $\sigma(A)$.

**Exercise 4** (10 Points)**.** Let $A, B \in B(\mathcal{H})$.

a) Show that the non-zero elements of $\sigma(AB)$ and of $\sigma(BA)$ are the same i.e,

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

b) Show that in the case $\dim \mathcal{H} < \infty$ we even have

$$\sigma(AB) = \sigma(BA).$$

c) Show with the help of an example that the statement of (b) is not true in general in the infinite-dimensional case.