SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Prof. Dr. Roland Speicher
M.Sc. Felix Leid

Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 8
Due on Mon 9.1.2023

Exercise 1 (10 Points)**.** Consider the Hilbert space $\mathcal{H} = l_2(\mathbb{N})$ of square-summable sequences, together with the canonical ONB $(e_n)_{n\in\mathbb{N}}$. For $A \in B(\mathcal{H})$ we define its matrix coefficients by $a_{mn} := \langle Ae_m, e_n\rangle$ for all $m, n \in \mathbb{N}$. The operator A is called *Hilbert-Schmidt operator* if $(a_{mn}) \in l_2(\mathbb{N} \times \mathbb{N})$, i.e., if

$$\|A\|_{\mathrm{HS}} := \left(\sum_{m,n=1}^{\infty} |a_{mn}|^2\right)^{1/2} < \infty.$$

The norm $\|\cdot\|_{\mathrm{HS}}$ is called *Hilbert-Schmidt norm*. The operator norm on $B(\mathcal{H})$ will be denoted by $\|\cdot\|_\infty$ in the following.

1. Show that for each infinite matrix $(a_{mn}) \in l_2(\mathbb{N} \times \mathbb{N})$ there is a bounded operator $A \in B(\mathcal{H})$ with matrix coefficients (a_{mn}); furthermore, we have $\|A\|_\infty \leq \|A\|_{\mathrm{HS}}$.
2. Show that each Hilbert-Schmidt operator is compact, but not every compact operator is a Hilbert-Schmidt operator.
3. Let A be a Hilbert-Schmidt operator and B and C bounded operators on $\mathcal{H}$. Show that BAC is a Hilbert-Schmidt operator and that we have

$$\|BAC\|_{\mathrm{HS}} \leq \|B\|_\infty \cdot \|A\|_{\mathrm{HS}} \cdot \|C\|_\infty.$$

[We say that the Hilbert-Schmidt operators are a *normed operator ideal* in $B(\mathcal{H})$.]

Exercise 2 (10 Points)**.** Let $V : C[0,1] \to C[0,1]$ be the Volterra operator on continuous functions, given by

$$(Vf)(t) := \int_0^t f(s)ds.$$

1. Show that V is a compact operator.
2. Show that V has no eigenvalues.
3. Show that the spectrum of V consists only of 0, $\sigma(V) = \{0\}$.

Exercise 3 (10 Points)**.** Prove the following complex version of the theorem of Stone-Weierstrass by reducing it to the real version from the video: Let K be a compact set and $C(K, \mathbb{C})$ the set of continuous functions $f : K \to \mathbb{C}$, equipped with the sup-norm. Let $A \subset C(K, \mathbb{C})$ be a unital subalgebra (i.e. $1 \in A$ and $f, g \in A$, $\alpha, \beta \in \mathbb{C}$ implies $\alpha f + \beta g, fg \in A$) which separates the points of K and which is closed under complex conjugation (i.e., $f \in A$ implies $\bar{f} \in A$). Then A is dense in $C(K, \mathbb{C})$.

Exercise 4 (10 Points)**.** Read about product topology and the Tychonoff theorem and discuss this in the tutorial. Does Tychonoff also work for sequentially compact?