SAARLAND UNIVERSITY
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 9
Due on Mon 16.1.2023

Exercise 1 (10 Points)**.** We define

$$l_1(\mathbb{Z}) := \left\{ (a_n)_{n\in\mathbb{Z}} \colon a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n| < \infty \right\}$$

and the *convolution* $*$ on $l_1(\mathbb{Z})$ by

$$(c_n)_{n\in\mathbb{Z}} := (a_n)_{n\in\mathbb{Z}} * (b_n)_{n\in\mathbb{Z}}, \qquad c_n := \sum_{k\in\mathbb{Z}} a_k b_{n-k}.$$

Show that $l_1(\mathbb{Z})$, equipped with $*$ as multiplication and

$$\|(a_n)_{n\in\mathbb{Z}}\|_1 := \sum_{n\in\mathbb{Z}} |a_n|$$

as norm, is a Banach algebra.

Exercise 2 (10 Points)**.** Let K be a compact topological space and $A = C(K)$ the Banach algebra of continuous functions on K (with point-wise operations as addition and multiplication and the with the sup-norm). Calculate the spectrum $\sigma(f)$ for $f \in C(K)$ and compare the spectral radius $r(f)$ with the norm $\|f\|_\infty$.

Exercise 3 (10 Points)**.** Let A be a Banach algebra without a unit (i.e. A satisfies all axioms of a Banach algebra but the ones that include the unit). Show that $A_1 := A \times \mathbb{C}$, together with the operations

$$\begin{aligned}
(a, \lambda) + (b, \mu) &:= (a + b, \lambda + \mu) \\
\mu(a, \lambda) &:= (\mu a, \mu\lambda) \\
(a, \lambda)(b, \mu) &:= (ab + \lambda b + \mu a, \lambda\mu) \\
\|(a, \lambda)\| &:= \|a\| + |\lambda|
\end{aligned}$$

for any $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$, is a Banach algebra with unit. What happens in this construction if A already has a unit? Does A_1 have two units? What happens when A is a C^*-algebra? Is A_1 a C^*-algebra as well?

Exercise 4 (10 Points)**.** The *one sided* shift $S\colon l_2 \to l_2$ is defined by $Se_n := e_{n+1}$, where $(e_n)_{n=1}^{\infty}$ is the standard ONB of l_2; compare Assignment 7, Exercise 1. Determine the spectrum of S and the spectrum of S^*.