SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 1
Due on Mon 7.11.2022, hand in before the lecture.

Exercise 1 (10 Points)**.** Let V be a pre-Hilbert space. Prove the following assertions:

a) Polarization Identity: We have for all $x, y \in V$

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2.$$

b) Parallelogram Law: We have for all $x, y \in V$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

c) Pythagorean Theorem: Let $x_1, \dots, x_n$ be pairwise orthogonal, then

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2.$$

Exercise 2 (10 Points)**.** Show that a normed space is a pre-Hilbert space if and only if its norm fulfills the Parallelogram Law.

Exercise 3 (10 Points)**.** We denote by $C[a, b]$ the set of continuous functions on the interval $[a, b]$.

a) Show that

$$\|f\|_{\max} := \max_{t \in [a,b]} |f(t)|,$$

defines a norm on $C[a, b]$.

b) Show that $(C[a, b], \|\cdot\|_{\max})$ is a Banach space.

c) Show that $(C[a, b], \|\cdot\|_{\max})$ is not a Hilbert space.

Exercise 4 (10 Points)**.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_0$ a closed linear subspace. Let $x \in \mathcal{H}$ and $x_0 \in \mathcal{H}_0$ such that

$$\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0).$$

Recall from the lecture that the latter condition on x_0 is equivalent to the condition that $x - x_0 \perp H_0$ and that x_0 is uniquely determined by this condition. Then we define $P\colon \mathcal{H} \to \mathcal{H}_0$ by $Px = x_0$. Show the following properties of P.

a) P is a linear mapping.

b) $\|Px\| \leq \|x\|$ for all $x \in \mathcal{H}$.

c) $P^2 = P$.

d) $\ker P = \mathcal{H}_0^\perp$ and $\operatorname{ran} P = \mathcal{H}_0$.