9. Adjoints of Operators on Banach Spaces

9.1. Motivation: For a Hilbert space $\mathcal{H}$ we have for $A \in B(\mathcal{H})$ an adjoint $A^* \in B(\mathcal{H})$, defined by

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x, y \in \mathcal{H}$$

Is there something similar for Banach spaces?

Corresponding to the inner product on $\mathcal{H}$ we have a duality

$$X \longleftrightarrow X^*$$

$$\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{C}$$

$$\langle x, x^* \rangle = x^*(x) \quad \text{for } x \in X, x^* \in X^*$$

Consider now $A \in B(X)$, $A : X \to X$

Then we want A^* with

$$\langle Ax, x^* \rangle \overset{!}{=} \langle x, A^* x^* \rangle \quad \forall \begin{array}{l} x \in X \\ x^* \in X^* \end{array}$$

thus $A^* : X^* \to X^*$

9.2. Proposition: Let X be a Banach space and $A \in B(X)$. Then there exists a uniquely determined operator $A^* \in B(X^*)$ s.th.

$$\langle Ax, x^* \rangle = \langle x, A^* x^* \rangle \quad \forall \begin{array}{l} x \in X \\ x^* \in X^* \end{array}$$

9.3 Def.: A^* is the adjoint of A.

Proof: i) Uniqueness: Assume $\forall x \in X, x^* \in X^*$

$$\langle x, A_1 x^* \rangle = \langle Ax, x^* \rangle = \langle x, A_2 x^* \rangle$$

i.e.: $\langle x, (A_1 - A_2) x^* \rangle = 0 \quad \forall x \in X$

i.e.: $[(A_1 - A_2) x^*](x) = 0 \quad$ -"-

i.e.: $(A_1 - A_2) x^* = 0 \quad \forall x^* \in X^*$

i.e.: $A_1 - A_2 = 0$, i.e. $A_1 = A_2$

ii) Existence: For $x^* \in X^*$ we define $A^* x^*$ by

$$A^* x^* (x) := x^* (Ax)$$

- linearity of A^* is clear
- boundedness follows from

$$\|A^* x^*\| = \sup_{\substack{x \in X \\ \|x\|=1}} |A^* x^*(x)|$$

$$= \sup_{\|x\|=1} \underbrace{|x^*(Ax)|}$$

$$\leq \|x^*\| \, \|Ax\|$$

$$\leq \|x^*\| \, \|A\| \, \underbrace{\|x\|}_{=1}$$

$$\leq \|A\| \cdot \|x^*\|$$

$\Rightarrow \|A^*\| \leq \|A\|$, thus $A^* \in B(X^*)$ 17

<u>9.4. Remark:</u> 1) How compares A to A^{**}?

$$\begin{array}{cccc} A^{**}: & X^{**} & \longrightarrow & X^{**} \\ & \cup & & \cup \\ & X & & X \end{array}$$

$$X \longrightarrow X^{**}$$
$$x \longmapsto \hat{x} \quad \text{with} \quad \hat{x}(x^*) = x^*(x)$$

Consider now $x \in X$, $x^* \in X^*$, then

$$\langle A^{**}(x), x^* \rangle = \langle x, A^* x^* \rangle = \langle Ax, x^* \rangle$$

$\Rightarrow A^{**}(x) = A(x) \quad \forall x \in X$

i.e. $A^{**}|_X = A$

more explicitly:

$$(A^{**}\hat{x})(x^*) = \hat{x}(A^*x^*)$$

$$= A^*x^*(x) = x^*(Ax) = \widehat{Ax}(x^*)$$

$$\Rightarrow A^{**}\hat{x} = \widehat{Ax}$$

i.e. $A^{**}x = Ax$ under identification
$x \leftrightarrow \hat{x}$

2) By proof of 9.2: $\|A^*\| \leq \|A\|$

hence also $\|A^{**}\| \leq \|A^*\|$

thus: $\|A\| = \|A^{**}|X\| \leq \|A^{**}\| \leq \|A^*\|$

$$\Rightarrow \|A\| = \|A^*\|$$

3) For a linear subspace $\mathcal{M} \subset X$ denote

$$\mathcal{M}^\perp := \{x^* \in X^* \mid x^*(x) = 0 \ \forall x \in \mathcal{M}\} \subset X^*$$

Then we have, as in Hilbert space case,

$$\ker A^* = (\operatorname{ran} A)^\perp.$$