The Theorem of Arzela - Ascoli

<u>Def</u>: Let X be a Banach space.
A set $K \subset X$ is <u>compact</u>, if every sequence in K has a subsequence which converges to a point in K

<u>Remarks</u>: 1) This is equivalent to a more general notion of compactness: Every open cover of K has a finite subcover.
The open definition is then called "sequentially compact". In normed spaces one has:

sequentially compact $\Leftrightarrow$ compact

2) K compact $\Rightarrow$ K closed

since: K is not closed

$\Rightarrow \exists\, x_n \in K,\ x_n \to x$ but $x \notin K$

$\Rightarrow (x_n)$ has no convergent subsequence with limit in K

(since any subsequence converges to $x \notin K$)

3) K compact $\Rightarrow$ K bounded

(i.e. $\exists\, M < \infty$:

$\|x\| \leq M \quad \forall x \in K$

since: K is not bounded

$\Rightarrow \exists\, x_n \in K$ s.th. $\|x_n\| \to \infty$

$\Rightarrow (x_n)$ has no convergent subsequence

(since: $x_{n(k)} \to x \Rightarrow \underbrace{\|x_{n(k)}\|}_{\to \infty} \to \|x\|$)

4) For finite-dimensional X we have

Heine-Borel: $K \subset X$ compact

$\Longleftrightarrow$ K is closed and bounded

[follows from Bolzano-Weierstraß]

5) In infinite dimensions this is not true any more; compactness is more than closed + bounded

e.g.: the closed unit ball is not compact in infinite dimensions

compactness $\hat{=}$ not enough space to keep so-many points separated from each other

in ∞ dimensions one has ∞-many directions to go

Consider now

$$X = C[0,1] = \{ f : [0,1] \to \mathbb{C} \mid f \text{ continuous} \}$$

X is infinite dimensional; e.g.,

x^n, $n \in \mathbb{N}_0$, are linearly independent

Note: $(x^n)_{n \in \mathbb{N}_0}$ cannot have a convergent subsequence (since $x^n \to \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ pointwise, thus only candidate for limit of any subsequence would be non-continuous, but uniform limit of continuous fcts must be continuous

<u>Motivation</u>: Consider

$$f_n \in \{ f \in C^1[0,1] \mid \|f\| \leq 1 \}$$

Just knowing this does not guarantee that the $\{f_n\}$ have to cluster somewhere. We need an additional reason for this, which keeps the $\{f_n\}$ closer together. In particular, this should be given for convergent sequences; so consider

$f_n \to f$; then

$$|f_n(x) - f_n(y)| \leq \underbrace{|f_n(x) - f(x)|}_{\substack{< \frac{\varepsilon}{3} \\ \text{for } n \geq N}} + \underbrace{|f(x) - f(y)|}_{\substack{< \frac{\varepsilon}{3} \\ \text{for} \\ |x-y| < \delta_f}} + \underbrace{|f(y) - f_n(y)|}_{\substack{< \frac{\varepsilon}{3} \\ \text{for } n \geq N}}$$

furthermore, we have additional, finitely many conditions

$$|f_1(x) - f_1(y)| < \varepsilon \qquad \text{for } |x-y| < \delta_1$$

$$\vdots$$

$$|f_{N-1}(x) - f_{N-1}(y)| < \varepsilon \qquad \text{for } |x-y| < \delta_N$$

thus we have for all m:

$$|f_m(x) - f_m(y)| < \varepsilon \qquad \forall |x-y| < \delta := \min(\delta_1, \dots, \delta_{N-1}, \delta_f)$$

$\uparrow$

continuity condition for the whole family (f_m) with the same δ for all elements in the family

$\rightarrow$ (f_n) is "equicontinuous"

"gleichgradig stetig"

<u>Def.</u>: A subset $A \subset C[0,1]$ is <u>equicontinuous</u> if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.th.

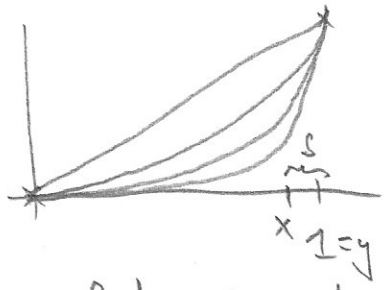
$$|f(x) - f(y)| < \varepsilon \qquad \forall |x-y| < \delta \quad \forall f \in A$$

[the same δ works for all $f \in A$]

"one δ to rule them all"

<u>Remark:</u> Note that $(x^n)_{n\geq 1}$ is not equicontinuous.

Take $\varepsilon = \frac{1}{2}$



(x^n) equicontinuous would imply:

there is $\delta > 0$ s.th. for $1-\delta \leq x \leq 1$ $(y=1)$

$$|x^n - 1^n| < \varepsilon = \frac{1}{2} \quad \text{for all } n$$

$$\begin{array}{ccc} \downarrow{\scriptstyle n\to\infty} & \| & \\ 0 & 1 & \text{contradiction} \end{array}$$

<u>Theorem of Arzela-Ascoli:</u> For $A \subset C^1[0,1]$ we have:

A compact $\Longleftrightarrow$ A closed, bounded and equicontinuous

<u>Proof:</u> "$\Longrightarrow$" "closed" and "bounded" follows in general from "compact"

Assume A is not equicontinuous, i.e.

$\exists\, \varepsilon > 0 \;\; \forall\, \delta = \frac{1}{n} \;\; \exists\, x_n, y_n$ with $|x_n - y_n| < \frac{1}{n}$
$\exists\, f_n \in A$

s.th. $|f_n(x_n) - f_n(y_n)| > \varepsilon$

$\Rightarrow$ no subsequence of (f_n) is equicontinuous

however: A compact $\Rightarrow \exists$ convergent subseq. of (f_n)

$\overset{\text{Motivation}}{\Longrightarrow}$ this subsequence is equicontinuous

contradiction, thus: A is equicontinuous

"$\Leftarrow$" Consider $(f_n)_{n\geq 1} \subset A$

to show: $\exists$ convergent subsequence

first we show existence of subsequence which converges pointwise at rational arguments

Let $x_1, x_2, x_3, \ldots$ be enumeration of

$$[0,1] \cap \mathbb{Q}$$

then: $(f_n(x_1))_n$ bounded sequence in $\mathbb{R}$

$\overset{\text{Bolzano-Weierstraß}}{\Longrightarrow}$ $\exists$ convergent subsequence $(f_{n(k)}(x_1))_k$

put $y_1 := \lim_{k\to\infty} f_{n(k)}(x_1)$

Now consider $(f_{n(k)}(x_2))_k$

this is bounded sequence in $\mathbb{R}$

$\overset{\text{Bolzano-}}{\underset{\text{Weierstraß}}{\Longrightarrow}}$ $\exists$ convergent subsequence $(f_{n(k(\ell))}(x_2))_\ell$

$$\text{put } y_2 := \lim_{\ell \to \infty} f_{n(k(\ell))}(x_2)$$

note that we still have

$$\lim_{\ell \to \infty} f_{n(k(\ell))}(x_1) = y_1$$

Rename now

$$f_r^{(0)} := f_r$$

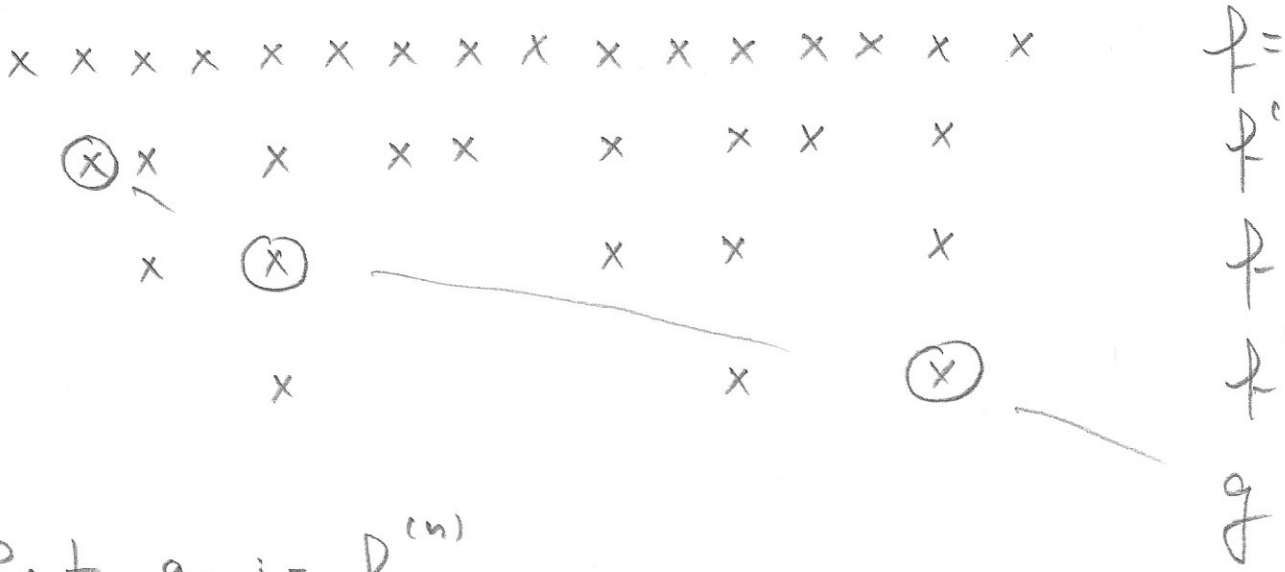
$$f_r^{(1)} := f_{n(r)}$$

$$f_r^{(2)} := f_{n(k(r))}$$

and continue in this way. This yields subsequences $(f_n^{(1)}), (f_n^{(2)}), \dots$ of (f_n) with

- $(f_n^{(k+1)})_n$ is subsequence of $(f_n^{(k)})$
- $\lim_{n \to \infty} f_n^{(k)}(x_i) = y_i$ for $i = 1, 2, \dots, k$

Each $f^{(k)}$ is good at finitely many $x_1, \dots, x_k$; we need a subsequence which is good at infinitely many $x_1, x_2, \dots$ $\rightsquigarrow$ diagonal argument



Put $g_n := f_n^{(n)}$

$\Rightarrow$ (g_n) is, for each k, eventually a subsequence of $(f_n^{(k)})$, i.e.

$$\forall k: \lim_{n\to\infty} g_n(x_k) = \lim_{n\to\infty} f_n^{(k)}(x_k) = y_k$$

$(g_n)_n$ is now our candidate for a convergent subsequence of the original sequence (f_n)

We know:

- (g_n) is subsequence of (f_n)
- $g_n(x)$ converges for all $x \in [0,1] \cap \mathbb{Q}$

We still have to show:

- $g_n(x)$ converges for all $x \in [0,1]$
- g_n converges uniformly

Fix $x \in [0,1]$; we will show that $(g_n(x))$ is a Cauchy sequence

$$|g_n(x) - g_m(x)| \leq \underbrace{|g_n(x) - g_n(x_i)|}_{\substack{<\varepsilon \\ \text{if } |x-x_i|<\delta \\ \text{where } \delta \text{ is} \\ \text{independent of } n, \\ \text{since } (g_n) \text{ equicont.} \\ (\text{as subsequence of } (f_n))}} + \underbrace{|g_n(x_i) - g_m(x_i)|}_{\substack{\text{fix } x_i \text{ with} \\ |x-x_i|<\delta, \\ \text{then } |\dots|<\varepsilon \\ \text{for } n,m \geq N, \\ \text{since } (g_n(x_i)) \\ \text{converges}}} + \underbrace{|g_m(x_i) - g_m(x)|}_{<\varepsilon}$$

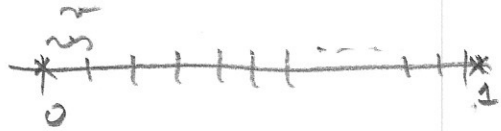
$\Rightarrow (g_n(x))_n$ converges

In order to get uniform convergence, the estimate above has to be independent of x; this is not true in this form, since N depends on chosen x_i

However, if we have a finite number of x_i ($i \in I$, $|I| < \infty$) to choose from, then we only have finitely many N_i and we can take the same $N := \max_{i \in I} N_i$ for all x

thus: we need finitely many $x_i \in \mathbb{Q}$ s.th.
for each $x \in [0,1]$ we have at least one
x_i with $|x - x_i| < \delta$ (for fixed δ)

but this is easy: just take $r \in \mathbb{Q}$ with
$0 < r < \delta$ and as x_i:
$r, 2r, 3r, \dots, \left(\frac{1}{r}\right) \cdot r$



$\Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}:$

$$|g_n(x) - g_m(x)| < 3\varepsilon \qquad \forall x \in [0,1] \quad \forall n,m \geq N$$

$\Rightarrow (g_n)_n$ converges uniformly. $\square$

<u>Example</u>: Consider $(f_n)_{n\in\mathbb{N}}$ in $C[0,1]$ with $\|f_n\| \leq 1$.

How can we ensure equicontinuity concretely?

Assume that all f_n' exist and that $\exists c > 0$

$\|f_n'\| \leq c \quad \forall n$

then: $|f_n(x) - f_n(y)| = |f_n'(\xi)(x-y)|$

mean value theorem

$\leq c|x-y|$

$\Rightarrow \{f_n\}_n$ equicontinuous, bounded

$\Rightarrow \overline{\{f_n\}_n}$ —— " —— + closed

thus compact

$\Rightarrow (f_n)_n$ has uniformly convergent subsequence!