7. The Baire Category Theorem and Applications

7.1. Theorem of Baire: If a Banach space X is the union of countably many closed subsets F_n, then at least one F_n has an interior point.

$$\left.\begin{array}{l} X = \bigcup_{n\in\mathbb{N}} F_n \\ \\ F_n \text{ closed} \end{array}\right\} \Rightarrow \exists k : \overset{\circ}{F}_k \neq \emptyset$$

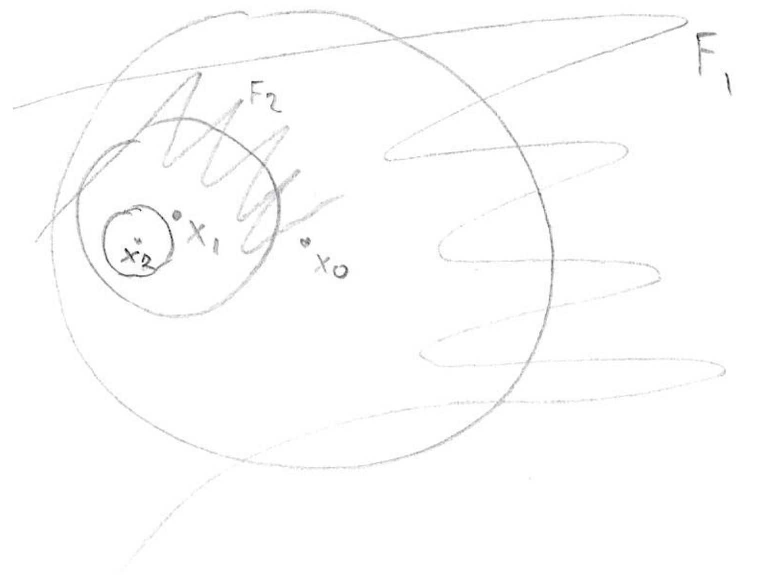
Proof: indirectly: Assume $\overset{\circ}{F}_k = \emptyset \quad \forall k$

we will show: $\exists x \in X$ s.th. $x \notin \bigcup_{n\in\mathbb{N}} F_n$

note: $\overset{\circ}{F}_k = \emptyset \Rightarrow \mathcal{U}_\varepsilon(x) \setminus F_k \neq \emptyset \quad \forall x \in X, \varepsilon > 0$

$\mathcal{U}_\varepsilon(x) := \{ y \in X \mid \|x-y\| < \varepsilon \}$

Idea: construct sequence $x_0, x_1, x_2, \dots$ s.th. $x_n \to x$ and $x \notin F_k \quad \forall k$



Choose $x_0 \in X$, $\varepsilon_0 > 0$ arbitrary

$\Rightarrow \mathcal{U}_{\varepsilon_0}(x_0) \setminus F_1 \neq \emptyset$ and open

$\Rightarrow \exists\, x_1 \in X$, $\varepsilon_1 > 0$ s.th. $\mathcal{U}_{\varepsilon_1}(x_1) \subset \mathcal{U}_{\varepsilon_0}(x) \setminus F_1$

(i.e. $\mathcal{U}_{\varepsilon_1}(x_1) \cap F_1 = \emptyset$)

we can choose ε_1 so that: $\varepsilon_1 \leq \frac{\varepsilon_0}{2}$ and

$\overline{\mathcal{U}_{\varepsilon_1}(x_1)} \cap F_1 = \emptyset$

in the same way: $\mathcal{U}_{\varepsilon_1}(x_1) \setminus F_2 \neq 0$ and open

$\Rightarrow \exists\, x_2 \in X$, $\varepsilon_2 > 0$: $\overline{\mathcal{U}_{\varepsilon_2}(x_2)} \subset \mathcal{U}_{\varepsilon_1}(x_1) \setminus F_2$

thus: $\overline{\mathcal{U}_{\varepsilon_2}(x_2)} \cap F_2 = \emptyset$

$\mathcal{U}_{\varepsilon_2}(x_2) \cap F_1 = \emptyset$ (since $\overline{\mathcal{U}_{\varepsilon_2}(x_2)} \subset \overline{\mathcal{U}_{\varepsilon_1}(x_1)}$)

without restriction: $\varepsilon_2 \leq \frac{\varepsilon_1}{2} \leq \frac{\varepsilon_0}{4}$

continue in this way: this gives sequence $(x_n)_{n \in \mathbb{N}}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ s.th.

$$\overline{\mathcal{U}_{\varepsilon_{n+1}}(x_{n+1})} \subset \mathcal{U}_{\varepsilon_n}(x_n) \setminus F_{n+1}$$

and $\varepsilon_{n+1} \leq \frac{\varepsilon_n}{2} \leq \frac{\varepsilon_0}{2^n}$

$\Rightarrow \|x_n - x_m\| \leq \varepsilon_n$ (let $m > n$), since $x_m \in \mathcal{U}_{\varepsilon_n}(x_n)$

$\leq \frac{\varepsilon_0}{2^n}$

$\to 0$ for n, m suff. large

$\Rightarrow (x_n)$ CS $\overset{X \text{ complete}}{\Longrightarrow} \exists x \in X$ s.th. $x_n \to x$

remains to show: $x \notin \bigcup F_n$

Fix k: $\quad x_n \in \mathcal{U}_{\varepsilon_k}(x_k) \quad \forall n \geq k$

$$\Rightarrow x = \lim_{n\to\infty} x_n \in \overline{\mathcal{U}_{\varepsilon_k}(x_k)} \subset \mathcal{U}_{\varepsilon_{k-1}}(x_{k-1}) \setminus F_k$$

$$\Rightarrow x \notin F_k \qquad \forall k$$

$$\Rightarrow x \notin \bigcup_{k\in\mathbb{N}} F_k$$

$\square$

7.2. Remarks: 1) Theorem is also true for complete metric spaces.

2) Let us define, for $F \subset X$:

F nowhere dense in X :$\Leftrightarrow$ $\overline{F}$ has no interior points

F is of first category in X :$\Leftrightarrow$ $F = \bigcup_{n\in\mathbb{N}} F_n$, all F_n nowhere dense

F is of second category in X :$\Leftrightarrow$ F is not of first category

In this language, 7.1. says:

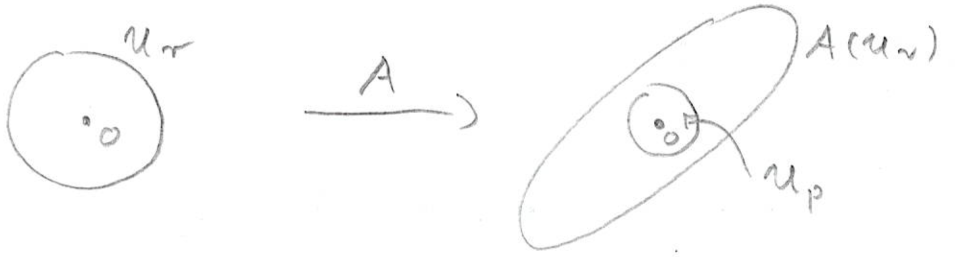
A Banach space (or a complete metric space) is of second category in itself.

In this form, 7.1. is called Baire Category Theorem.

7.3. Open Mapping Theorem: Let X, Y be Banach spaces and $A: X \to Y$ a continuous linear mapping onto Y (i.e. A is surjective). Then A is an open mapping, i.e. for each open set $\mathcal{U} \subset X$ its image $A(\mathcal{U}) \subset Y$ is an open set, too.

Proof: Main step is to prove that A is open at 0, i.e.

(*) For each ball $\mathcal{U}_r(0) \subset X$ there is a ball $\mathcal{U}_p(0) \subset Y$ s.th. $\mathcal{U}_p(0) \subset A(\mathcal{U}_r(0))$



Assume (*) is shown: Consider $\mathcal{U} \subset X$ open and $y \in A(\mathcal{U})$, i.e. $y = Ax$ for some $x \in \mathcal{U}$

$\Rightarrow \exists\, \mathcal{U}_r(0) \subset X$ s.th. $\underbrace{x + \mathcal{U}_r(0)}_{\mathcal{U}_r(x)} \subset \mathcal{U}$

$\overset{(*)}{\Rightarrow} \exists\, \mathcal{U}_p(0) \subset Y$ s.th. $\mathcal{U}_p(0) \subset A(\mathcal{U}_r(0))$

$$\Rightarrow \mathcal{U}_p(y) = y + \mathcal{U}_p(0) \subset Ax + A(\mathcal{U}_r(0)) = A(x + \mathcal{U}_r(0)) \subset A(\mathcal{U})$$

thus $A(\mathcal{U})$ open

We prove (*) by showing

(i) (*) in the weaker form $\mathcal{U}_p(0) \subset \overline{A(\mathcal{U}_r(0))}$

(ii) We always have: $\overline{A(\mathcal{U}_{r/2}(0))} \subset A(\mathcal{U}_r(0))$

(i) A surjective $\Rightarrow$ $Y = \bigcup_{k=1}^{\infty} \overline{A(\mathcal{U}_{\frac{kr}{2}}(0))}$

$$= \bigcup_{k=1}^{\infty} k\,\overline{A(\mathcal{U}_{r/2}(0))}$$

$\overset{\text{Baire}}{\underset{7.1}{\Rightarrow}}$ $\exists k$ s.th. $k\overline{A(\mathcal{U}_{r/2}(0))}$ contains interior point y_0, i.e.

$\exists s$: $\mathcal{U}_s(y_0) \subset k\overline{A(\mathcal{U}_{r/2}(0))}$

$\Rightarrow \mathcal{U}_{s/k}(y_0/k) \subset \overline{A(\mathcal{U}_{r/2}(0))}$

$\frac{s}{k} \rightsquigarrow s$, $\frac{y_0}{k} \rightsquigarrow y_0$

i.e. $\mathcal{U}_s(y_0) \subset \overline{A(\mathcal{U}_{r/2}(0))}$

we will show: $\mathcal{U}_s(0) \subset \overline{A(\mathcal{U}_r(0))}$

$y_0 \in \overline{A(\mathcal{U}_{r/2}(0))}$ $\Rightarrow$ $\exists x_n \in \mathcal{U}_{r/2}(0)$ s.th. $A(x_n) \to y_0$

Consider $y \in \mathcal{U}_s(0) \Rightarrow y_0 + y \in \mathcal{U}_s(y_0) \subset \overline{A(\mathcal{U}_{r/2}(0))}$

$\Rightarrow \exists z_n \in \mathcal{U}_{r/2}(0)$ s.th. $A(z_n) \to y_0 + y$

$\Rightarrow A(z_n - x_n) \to y$ and $z_n - x_n \in \mathcal{U}_r(0)$

$\Rightarrow y \in \overline{A(\mathcal{U}_r(0))}$

(ii) By (i), there are $p_n > 0$ s.th.

$$\mathcal{U}_{p_n}(0) \subset \overline{A(\mathcal{U}_{r/2^n}(0))}$$

we can assume $p_n \to 0$

Fix now $y \in \overline{A(\mathcal{U}_{r/2}(0))}$, to show: $y \in A(\mathcal{U}_r(0))$

$y \in \overline{A(\mathcal{U}_{r/2}(0))} \Rightarrow \exists\, y_1 \in A(\mathcal{U}_{r/2}(0))$ s.th $\|y - y_1\| < p_2$

i.e. $y_1 = Ax_1$ and $\|x_1\| < \frac{r}{2}$

$y - y_1 \in \mathcal{U}_{p_2}(0) \subset \overline{A(\mathcal{U}_{r/4}(0))}$

$\Rightarrow \exists\, y_2 \in A(\mathcal{U}_{r/4}(0))$ s.th $\|y - y_1 - y_2\| < p_3$

i.e. $y_2 = Ax_2$ and $\|x_2\| < \frac{r}{4}$

continue in this way:

$\exists\, y_n \in A(\mathcal{U}_{r/2^n}(0))$ s.th. $\|y - y_1 - y_2 \dots - y_n\| < p_{n+1}$

i.e. $y_n = Ax_n$ and $\|x_n\| < \frac{r}{2^n}$

thus: $y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} Ax_n$

moreover: $x := \sum_{n=1}^{\infty} x_n$ exists, since $\|x_n\| < \frac{r}{2^n}$

and $\|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \frac{r}{2^n} = r$

$\Rightarrow x \in \mathcal{U}_r(0)$

A continuous $\Rightarrow Ax = A\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} Ax_n = y$

$\Rightarrow y \in A(\mathcal{U}_r(0))$

7.4. Corollary (Inverse Mapping Theorem): Let X and Y be Banach spaces and $A : X \to Y$ a bounded linear transformation that is bijective. Then $A^{-1} : Y \to X$ is bounded.

Proof: A bijective $\Rightarrow$ A^{-1} exists

7.3. $\Rightarrow$ A open $\Leftrightarrow$ A^{-1} continuous

7.5. Remark: Another formulation of 7.4. goes like this: A continuous linear bijection between Banach spaces is an isomorphism.

($\|Ax\| \leq d\|x\| \ \forall x \in X$, A bijective

$\Rightarrow \exists c > 0 : \|Ax\| \geq c\|x\| \quad \forall x \in X$)

7.6 Theorem (Principle of Uniform Boundedness): Let X be a Banach space and Y a normed space. Let $\mathcal{A} \subseteq B(X, Y)$ such that for each $x \in X$:

$$\sup_{A \in \mathcal{A}} \|Ax\| < \infty.$$

Then $$\sup_{A \in \mathcal{A}} \|A\| < \infty$$

Proof: Put $B_n := \{x \in X \mid \|Ax\| \leq n \quad \forall A \in \mathcal{A}\}$ closed

assumption $\Rightarrow X = \bigcup_{n=1}^{\infty} B_n$

$\overset{\text{Baire}}{\underset{7.1}{\Longrightarrow}} \exists k : \overset{\circ}{B_k} \neq \emptyset$, i.e. $\exists x \in X, \varepsilon > 0 : \mathcal{U}_\varepsilon(x) \subset B_k$

we will show: $\mathcal{U}_\varepsilon(0) \subset B_k$

note: • $y \in B_k \Rightarrow -y \in B_k$

• $\left.\begin{array}{l} y_1, y_2 \in B_k \\ 0 \le t \le 1 \end{array}\right\} \Rightarrow t y_1 + (1-t) y_2 \in B_k$ (i.e. B_k convex)

(since: $\|A(t y_1 + (1-t) y_2)\| \le t \underbrace{\|A y_1\|}_{\le n} + (1-t) \underbrace{\|A y_2\|}_{\le n} \le n$)

now: $\mathcal{U}_\varepsilon(-x) = -\mathcal{U}_\varepsilon(x) \subset B_k$

thus: $\mathcal{U}_\varepsilon(0) \subset \frac{1}{2}\mathcal{U}_\varepsilon(x) + \frac{1}{2}\mathcal{U}_\varepsilon(-x)$ $\quad$ ($\underset{\mathcal{U}_\varepsilon(0)}{\underset{\cap}{y}} = \frac{1}{2}\underset{\mathcal{U}_\varepsilon(x)}{\underset{\cap}{(y+x)}} + \frac{1}{2}\underset{\mathcal{U}_\varepsilon(-x)}{\underset{\cap}{(y-x)}}$)

$\subset \frac{1}{2} B_k + \frac{1}{2} B_k$

$\subset B_k$

hence: $\mathcal{U}_\varepsilon(0) \subset B_k \Rightarrow \overline{\mathcal{U}_\varepsilon(0)} \subset B_k$ (since B_k closed)

but this means:

$\|x\| \leqslant \varepsilon \Rightarrow \|Ax\| \le k \quad \forall A \in \mathcal{A}$

$\Rightarrow \|A\| = \sup\limits_{\|x\| = \varepsilon} \frac{\|Ax\|}{\|x\|} \le \frac{k}{\varepsilon} \quad \forall A \in \mathcal{A}$

$\Rightarrow \sup\limits_{A \in \mathcal{A}} \|A\| \le \frac{k}{\varepsilon} < \infty$

□

7.7. Corollary (Banach-Steinhaus Theorem): Let X and Y be Banach spaces and $(A_n)_{n\in\mathbb{N}}$ a sequence in $B(X,Y)$ that converges pointwise, i.e.

$\forall x \in X \ \exists y \in Y$ s.th. $\|A_n x - y\| \to 0$.

Define a linear operator A by

$$Ax := \lim_{n\to\infty} A_n x \quad (x \in X)$$

Then A is bounded and $\|A\| \le \sup_n \|A_n\| < \infty$

Proof: linearity of A clear

$x \in X \overset{\text{assumpt}}{\Longrightarrow} (A_n x)$ converges

$\Rightarrow \sup_n \|A_n x\| < \infty \quad \forall x \in X$

$\overset{7.6}{\Longrightarrow} S := \sup_n \|A_n\| < \infty$

Consider $x \in X$, $\|x\| \le 1$

$$\Rightarrow \|Ax\| \le \underbrace{\|Ax - A_n x\|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\|A_n x\|}_{\le S}$$

$$\le \varepsilon + S \quad \text{for arbitr. } \varepsilon > 0$$

$\overset{\varepsilon \searrow 0}{\Longrightarrow} \|Ax\| \le S$

$\Rightarrow \|A\| \le S = \sup \|A_n\| < \infty$

$\Rightarrow A \in B(X,Y)$ $\square$