

7. The Baire Category Theorem and Applications

(7-1)

7.1. Theorem of Baire: If a Banach space X is the union of countably many closed subsets F_n , then at least one F_n has an interior point.

$$\left. \begin{array}{l} X = \bigcup_{n \in \mathbb{N}} F_n \\ F_n \text{ closed} \end{array} \right\} \Rightarrow \exists k : \overset{\circ}{F}_k \neq \emptyset$$

Proof: indirectly: Assume $\overset{\circ}{F}_k = \emptyset \quad \forall k$

We will show: $\exists x \in X$ s.t.h. $x \notin \bigcup_{n \in \mathbb{N}} F_n$

note: $\overset{\circ}{F}_k = \emptyset \Rightarrow U_\varepsilon(x) \setminus F_k \neq \emptyset \quad \forall x \in X, \varepsilon > 0$

$$\uparrow U_\varepsilon(x) := \{y \in X \mid \|x - y\| < \varepsilon\}$$

Idea: construct sequence $x_0, x_1, x_2, \dots$ s.t.h. $x_n \rightarrow x$

and $x \notin F_k \quad \forall k$



Choose $x_0 \in X$, $\varepsilon_0 > 0$ arbitrary

7-2

$\Rightarrow U_{\varepsilon_0}(x_0) \setminus F_1 \neq \emptyset$ and open

$\Rightarrow \exists x_1 \in X$, $\varepsilon_1 > 0$ s.t.h. $U_{\varepsilon_1}(x_1) \subset U_{\varepsilon_0}(x_0) \setminus F_1$
(i.e. $U_{\varepsilon_1}(x_1) \cap F_1 = \emptyset$)

We can choose ε_1 so that $\varepsilon_1 \leq \frac{\varepsilon_0}{2}$ and

$$\overline{U_{\varepsilon_1}(x_1)} \cap F_1 = \emptyset$$

in the same way: $U_{\varepsilon_1}(x_1) \setminus F_2 \neq \emptyset$ and open

$\Rightarrow \exists x_2 \in X$, $\varepsilon_2 > 0$: $\overline{U_{\varepsilon_2}(x_2)} \subset U_{\varepsilon_1}(x_1) \setminus F_2$

$$\text{thus: } \overline{U_{\varepsilon_2}(x_2)} \cap F_2 = \emptyset$$

$$U_{\varepsilon_2}(x_2) \cap F_1 = \emptyset \quad (\text{since } \overline{U_{\varepsilon_2}(x_2)} \subset \overline{U_{\varepsilon_1}(x_1)})$$

$$\text{without restriction: } \varepsilon_2 \leq \frac{\varepsilon_1}{2} \leq \frac{\varepsilon_0}{4}$$

continue in this way: this gives sequence $(x_n)_{n \in \mathbb{N}}$

and $(\varepsilon_n)_{n \in \mathbb{N}}$ s.t.h.

$$\overline{U_{\varepsilon_{n+1}}(x_{n+1})} \subset U_{\varepsilon_n}(x_n) \setminus F_{n+1}$$

$$\text{and } \varepsilon_{n+1} \leq \frac{\varepsilon_n}{2} \leq \frac{\varepsilon_0}{2^n}$$

$$\Rightarrow \|x_n - x_m\| \leq \varepsilon_n \quad (\text{let } m > n), \text{ since } x_m \in U_{\varepsilon_n}(x_n)$$

$$\leq \frac{\varepsilon_0}{2^n}$$

$\rightarrow 0$ for n, m suff. large

$$\Rightarrow (x_n) \text{ CS} \stackrel{x \text{ complete}}{=} \exists x \in X \text{ s.t. } x_n \rightarrow x$$

2-3

remains to show: $x \notin \bigcup F_n$

$$\text{Fix } k : x_n \in U_{\varepsilon_k}(x_k) \quad \forall n \geq k$$

$$\Rightarrow x = \lim_{n \rightarrow \infty} x_n \in \overline{U_{\varepsilon_k}(x_k)} \subset U_{\varepsilon_{k-1}}(x_{k-1}) \setminus F_k$$

$$\Rightarrow x \notin F_k \quad \forall k$$

$$\Rightarrow x \notin \bigcup_{k \in \mathbb{N}} F_k$$

□

7.2. Remarks: 1) Theorem is also true for complete metric spaces.

2) Let us define, for $F \subset X$:

F nowhere dense in $X : \Leftrightarrow \overline{F}$ has no interior points

F is of first category in $X : \Leftrightarrow F = \bigcup_{n \in \mathbb{N}} F_n$, all F_n nowhere dense

F is of second category in $X : \Leftrightarrow F$ is not of first category

In this language, 7.1. says:

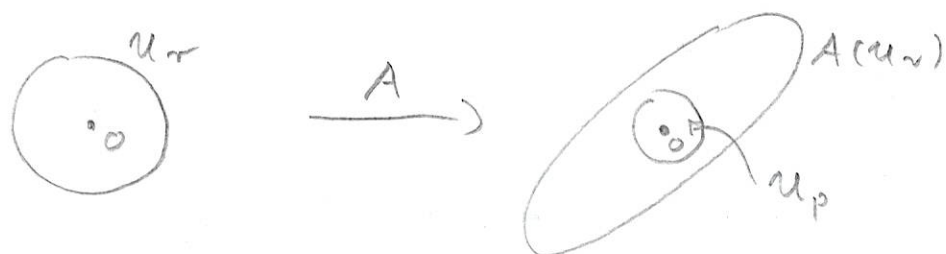
A Banach space (or a complete metric space) is of second category in itself.

In this form, 7.1. is called Baire Category Theorem.

7.3. Open Mapping Theorem: Let X, Y be Banach spaces (7-4) and $A: X \rightarrow Y$ a continuous linear mapping onto Y (i.e. A is surjective). Then A is an open mapping, i.e. for each open set $U \subset X$ its image $A(U) \subset Y$ is an open set, too.

Proof: Main step is to prove that A is open at 0 , i.e.

(*) For each ball $U_r(0) \subset X$ there is a ball $U_p(0) \subset Y$ s.th. $U_p(0) \subset A(U_r(0))$



Assume (*) is shown: Consider $U \subset X$ open

and $y \in A(U)$, i.e. $y = Ax$ for some $x \in U$

$\Rightarrow \exists U_r(0) \subset X$ s.th. $\underbrace{x + U_r(0)}_{U_r(x)} \subset U$

$\stackrel{(*)}{\Rightarrow} \exists U_p(0) \subset Y$ s.th. $U_p(0) \subset A(U_r(0))$

$\Rightarrow U_p(y) = y + U_p(0) \subset Ax + A(U_r(0))$
 $= A(x + U_r(0))$
 $\subset A(U)$

thus $A(U)$ open

We prove (*) by showing

(i) (*) in the weaker form $U_p(0) \subset \overline{A(U_n(0))}$

(ii) We always have: $\overline{A(U_{n/2}(0))} \subset \overline{A(U_n(0))}$

$$\begin{aligned} \text{(i) } A \text{ surjective } \Rightarrow Y &= \bigcup_{k=1}^{\infty} \overline{A(U_{\frac{k+r}{2}}(0))} \\ &= \bigcup_{k=1}^{\infty} k \overline{A(U_{n/2}(0))} \end{aligned}$$

$\xRightarrow[\text{7.1}]{\text{Baire}}$ $\exists k$ s.th. $k \overline{A(U_{n/2}(0))}$ contains interior point y_0 , i.e.

$$\exists s, U_s(y_0) \subset k \overline{A(U_{n/2}(0))}$$

$$\Rightarrow U_{s/k}(y_0/k) \subset \overline{A(U_{n/2}(0))}$$

$$\frac{s}{k} \rightsquigarrow s, \frac{y_0}{k} \rightsquigarrow y_0$$

$$\text{i.e. } U_s(y_0) \subset \overline{A(U_{n/2}(0))}$$

$$\text{we will show: } U_s(0) \subset \overline{A(U_n(0))}$$

$$y_0 \in \overline{A(U_{n/2}(0))} \Rightarrow \exists x_n \in U_{n/2}(0) \text{ s.th. } A(x_n) \rightarrow y_0$$

$$\text{Consider } y \in U_s(0) \Rightarrow y_0 + y \in U_s(y_0) \subset \overline{A(U_{n/2}(0))}$$

$$\Rightarrow \exists z_n \in U_{n/2}(0) \text{ s.th. } A(z_n) \rightarrow y_0 + y$$

$$\Rightarrow A(z_n - x_n) \rightarrow y \quad \text{and } z_n - x_n \in U_n(0)$$

$$\Rightarrow y \in \overline{A(U_n(0))}$$

(ii) By (i), there are $p_n > 0$ s.t.

$$U_{p_n}(0) \subset \overline{A(U_{r/2^n}(0))}$$

we can assume $p_n \rightarrow 0$

Fix now $y \in \overline{A(U_{r/2}(0))}$, to show: $y \in A(U_r(0))$

$$y \in \overline{A(U_{r/2}(0))} \Rightarrow \exists y_1 \in A(U_{r/2}(0)) \text{ s.t. } \|y - y_1\| < p_2$$

$$\text{i.e. } y_1 = Ax_1 \text{ and } \|x_1\| < \frac{r}{2}$$

$$y - y_1 \in U_{p_2}(0) \subset \overline{A(U_{r/4}(0))}$$

$$\Rightarrow \exists y_2 \in A(U_{r/4}(0)) \text{ s.t. } \|y - y_1 - y_2\| < p_3$$

$$\text{i.e. } y_2 = Ax_2 \text{ and } \|x_2\| < \frac{r}{4}$$

continue in this way:

$$\exists y_n \in A(U_{r/2^n}(0)) \text{ s.t. } \|y - y_1 - y_2 - \dots - y_n\| < p_{n+1}$$

$$\text{i.e. } y_n = Ax_n \text{ and } \|x_n\| < \frac{r}{2^n}$$

$$\text{thus: } y = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} Ax_n$$

$$\text{moreover: } x := \sum_{n=1}^{\infty} x_n \text{ exists, since } \|x_n\| < \frac{r}{2^n}$$

$$\text{and } \|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \frac{r}{2^n} = r$$

$$\Rightarrow x \in U_r(0)$$

$$A \text{ continuous} \Rightarrow Ax = A\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} Ax_n = y$$

$$\Rightarrow y \in A(U_r(0))$$

7.4. Corollary (Inverse Mapping Theorem): Let X and Y 7.3

be Banach spaces and $A: X \rightarrow Y$ a bounded linear transformation that is bijective. Then $A^{-1}: Y \rightarrow X$ is bounded.

Proof: A bijective $\Rightarrow A^{-1}$ exists

7.3. $\Rightarrow A$ open $\Leftrightarrow A^{-1}$ continuous

7.5. Remark: Another formulation of 7.4. goes like this:

A continuous linear bijection between Banach spaces is an isomorphism.

($\|Ax\| \leq d\|x\| \forall x \in X$, A bijective

$\Rightarrow \exists c > 0 : \|Ax\| \geq c\|x\| \forall x \in X$)

7.6 Theorem (Principle of Uniform Boundedness): Let X

be a Banach space and Y a normed space. Let

$\mathcal{A} \subseteq \mathcal{B}(X, Y)$ such that for each $x \in X$:

$$\sup_{A \in \mathcal{A}} \|Ax\| < \infty.$$

Then $\sup_{A \in \mathcal{A}} \|A\| < \infty$

Proof: Put $B_n := \{x \in X \mid \|Ax\| \leq n \quad \forall A \in \mathcal{A}\}$ closed

assumption $\Rightarrow X = \bigcup_{n=1}^{\infty} B_n$

$\stackrel{\text{Baire}}{\Rightarrow} \exists k : B_k \neq \emptyset$, i.e. $\exists x \in X, \varepsilon > 0 : \mathcal{U}_{\varepsilon}(x) \subset B_k$
7.1

we will show: $\mathcal{U}_{\varepsilon}(0) \subset B_k$