7.4. Corollary (Inverse Mapping Theorem): Let X and Y be Banach spaces and $A : X \to Y$ a bounded linear transformation that is bijective. Then $A^{-1} : Y \to X$ is bounded.

Proof: A bijective $\Rightarrow$ A^{-1} exists

7.3. $\Rightarrow$ A open $\Leftrightarrow$ A^{-1} continuous

7.5. Remark: Another formulation of 7.4. goes like this: A continuous linear bijection between Banach spaces is an isomorphism.

$(\|Ax\| \leq d \|x\| \ \forall x \in X$, A bijective

$\Rightarrow \exists c > 0 : \|Ax\| \geq c \|x\| \quad \forall x \in X)$

7.6 Theorem (Principle of Uniform Boundedness): Let X be a Banach space and Y a normed space. Let $\mathcal{A} \subseteq B(X, Y)$ such that for each $x \in X$:

$$\sup_{A \in \mathcal{A}} \|Ax\| < \infty .$$

Then $\sup_{A \in \mathcal{A}} \|A\| < \infty$

Proof: Put $B_n := \{ x \in X \mid \|Ax\| \leq n \quad \forall A \in \mathcal{A} \}$ closed

assumption $\Rightarrow$ $X = \bigcup_{n=1}^{\infty} B_n$

$\overset{\text{Baire}}{\underset{7.1}{\Rightarrow}}$ $\exists k : \mathring{B}_k \neq \emptyset$, i.e. $\exists x \in X, \varepsilon > 0 : \mathcal{U}_\varepsilon(x) \subset B_k$

we will show: $\mathcal{U}_\varepsilon(0) \subset B_k$

note: • $y \in B_k \Rightarrow -y \in B_k$

• $\left.\begin{array}{l} y_1, y_2 \in B_k \\ 0 \leq t \leq 1 \end{array}\right\} \Rightarrow t y_1 + (1-t) y_2 \in B_k$ (i.e. B_k convex)

(since: $\|A(t y_1 + (1-t) y_2)\| \leq t \underbrace{\|A y_1\|}_{\leq n} + (1-t) \underbrace{\|A y_2\|}_{\leq n} \leq n$)

now: $\mathcal{U}_\varepsilon(-x) = -\mathcal{U}_\varepsilon(x) \subset B_k$

thus: $\mathcal{U}_\varepsilon(0) \subset \frac{1}{2} \mathcal{U}_\varepsilon(x) + \frac{1}{2} \mathcal{U}_\varepsilon(-x)$ $\quad$ ($\underset{\mathcal{U}_\varepsilon(0)}{\underset{\cap}{y}} = \frac{1}{2}\underset{\mathcal{U}_\varepsilon(x)}{\underset{\cap}{(y+x)}} + \frac{1}{2}\underset{\mathcal{U}_\varepsilon(-x)}{\underset{\cap}{(y-x)}}$

$\subset \frac{1}{2} B_k + \frac{1}{2} B_k$

$\subset B_k$

hence: $\mathcal{U}_\varepsilon(0) \subset B_k \Rightarrow \overline{\mathcal{U}_\varepsilon(0)} \subset B_k$ (since B_k closed)

but this means:

$\|x\| \leq \varepsilon \Rightarrow \|Ax\| \leq k \quad \forall A \in \mathcal{A}$

$\Rightarrow \|A\| = \sup_{\|x\| = \varepsilon} \frac{\|Ax\|}{\|x\|} \leq \frac{k}{\varepsilon} \quad \forall A \in \mathcal{A}$

$\Rightarrow \sup_{A \in \mathcal{A}} \|A\| \leq \frac{k}{\varepsilon} < \infty$

$\square$

7.7. Corollary (Banach-Steinhaus Theorem): Let X and Y be Banach spaces and $(A_n)_{n\in\mathbb{N}}$ a sequence in $B(X,Y)$ that converges pointwise, i.e.

$$\forall x \in X \ \exists y \in Y \text{ s.th. } \|A_n x - y\| \to 0.$$

Define a linear operator A by

$$Ax := \lim_{n\to\infty} A_n x \quad (x \in X)$$

Then A is bounded and $\|A\| \leq \sup_n \|A_n\| < \infty$

Proof: linearity of A clear

$$x \in X \overset{\text{assumpt}}{\Longrightarrow} (A_n x) \text{ converges}$$

$$\Rightarrow \sup_n \|A_n x\| < \infty \qquad \forall x \in X$$

$$\overset{7.6}{\Longrightarrow} S := \sup_n \|A_n\| < \infty$$

Consider $x \in X$, $\|x\| \leq 1$

$$\Rightarrow \|Ax\| \leq \underbrace{\|Ax - A_n x\|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\|A_n x\|}_{\leq S}$$

$$\leq \varepsilon + S \qquad \text{for arbitr. } \varepsilon > 0$$

$$\overset{\varepsilon \searrow 0}{\Longrightarrow} \|Ax\| \leq S$$

$$\Rightarrow \|A\| \leq S = \sup \|A_n\| < \infty$$

$$\Rightarrow A \in B(X,Y)$$