12. Banach Algebras

in the following $\mathbb{F} = \mathbb{C}$!

12.1. Def.: 1) An algebra is a vector space A with a multiplication $\cdot : A \times A \to A$, such that $(A, +, \cdot)$ is a ring with unit 1 and

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \forall a, b \in A, \ \lambda \in \mathbb{C}$$

If $ab = ba$ $\forall a, b \in A$, then A is commutative.

2) A normed algebra is an algebra A, equipped with a norm $\|\cdot\|$, such that

i) $\|ab\| \leq \|a\| \cdot \|b\| \quad \forall a, b \in A$

ii) $\|1\| = 1$

3) A Banach algebra is a normed algebra which is complete.

4) A <u>C^*-algebra</u> is a Banach algebra with a mapping $*: A \to A$ such that:

i) $*$ is an involution, i.e. for $a, b \in A$, $\lambda \in \mathbb{C}$

$$(a+b)^* = a^* + b^*$$
$$(\lambda a)^* = \bar{\lambda}\, a^*$$
$$(ab)^* = b^* a^*$$
$$(a^*)^* = a$$

ii) $\| a^* a \| = \| a \|^2 \quad \forall a \in A$

5) Let A_1, A_2 be Banach algebras and $\varphi: A_1 \to A_2$ a linear map.

If φ is multiplicative, i.e,

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in A_1,$$

then φ is an (algebra) <u>homomorphism</u>.

If φ is bijective, then it is an (algebra) <u>isomorphism</u>.

If furthermore $\|\varphi(a)\| = \|a\| \ \forall a \in A_1$ then φ is an <u>isometric isomorphism</u>.

6) Let A_1, A_2 be C^*-algebras and $\varphi: A_1 \to A_2$ a homomorphism. If

$$\varphi(a^*) = \varphi(a)^* \quad \forall a \in A_1$$

then φ is a <u>$*$-homomorphism</u>.

<u>12.2. Remark:</u> One can also consider Banach algebras without unit. Those can be embedded in Banach algebras with unit. (Exercise!)

<u>12.3. Examples:</u> 1) $K \neq \emptyset$ compact space

$$C(K) := \{ f: K \to \mathbb{C} \mid f \text{ continuous} \}$$

is Banach algebra with

$$\|f\| := \sup_{t \in K} |f(t)|$$

$$\left.\begin{aligned} (f+g)(t) &:= f(t) + g(t) \\ (\lambda \cdot f)(t) &:= \lambda f(t) \\ (f \cdot g)(t) &:= f(t) \cdot g(t) \end{aligned}\right\} \quad \begin{aligned} f, g &\in C(K) \\ \lambda &\in \mathbb{C} \end{aligned}$$

$C(K)$ is a <u>C^*-algebra</u> with

$$f^*(t) := \overline{f(t)}$$

$C(K)$ is commutative

2) Let X be Banach space. Then $B(X)$ is a Banach algebra; it is non-commutative, if $\dim X > 1$

3) Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a C^*-algebra.

12.4. Lemma: 1) Let A be a normed algebra. Then the multiplication is continuous.

2) Let A be a C^*-algebra. Then

i) $\|a^*\| = \|a\| \quad \forall a \in A$

ii) $1^* = 1, \quad 0^* = 0$

Proof: 1) $\|ab - cd\| = \|a(b-d) + (a-c)d\|$

$$\leq \|a\| \|b-d\| + \|a-c\| \|d\|$$

thus: $\left.\begin{array}{l} a_n \to a \\ b_n \to b \end{array}\right\} \Rightarrow a_n b_n \longrightarrow ab$

2) i) $\|a\|^2 = \|aa^*\| \leq \|a\| \cdot \|a^*\|$

$\Rightarrow \|a\| \leq \|a^*\| \qquad$ (assume w.v. $a \neq 0$)

$\overset{a \leadsto a^*}{\Rightarrow} \|a^*\| \leq \|a^{**}\| = \|a\|$

$\Rightarrow \|a\| = \|a^*\|$

ii) $0^* = (0+0)^* = 0^* + 0^* \Rightarrow 0^* = 0$

$$1^* = 1 \cdot 1^*$$

$$\Rightarrow 1 = 1^{**} = (1 \cdot 1^*)^* = 1^{**} \cdot 1^* = 1 \cdot 1^* = 1^*$$

□

<u>12.5. Def.:</u> Let A be a Banach algebra.

1) $a \in A$ is <u>invertible</u>, if there exists $b \in A$ such that $a \cdot b = 1 = b \cdot a$

We denote b then by a^{-1}.

(Note: b is uniquely determined.)

$\mathcal{G}_A := \{ a \in A \mid a \text{ is invertible} \}$

is the <u>group of invertible elements</u>.

(group w.r.t. multiplication:

$x, y \in \mathcal{G}_A \Rightarrow (xy)^{-1} = y^{-1} x^{-1} \Rightarrow xy \in \mathcal{G}_A$)

2) $\varrho(a) := \{ \lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ is invertible} \}$

is the <u>resolvent set</u> of a. The function

$R(\cdot, a) : \varrho(a) \to A$, $R(\lambda, a) := (\lambda 1 - a)^{-1}$

$=: R(\lambda)$

is the <u>resolvent</u> of a

(in physics also: Greens function $G(z)$)

The complement of $\varrho(a)$,

$$\sigma(a) := \mathbb{C} \setminus \varrho(a)$$
$$= \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ not invertible}\}$$

is the <u>spectrum</u> of a.

<u>12.6. Lemma:</u> Let A be a Banach algebra and $x \in A$ with $\|x\| < 1$. Then $1 - x$ is invertible and we have

$$\|(1-x)^{-1}\| \leq \frac{1}{1 - \|x\|}$$

$$\|1 - (1-x)^{-1}\| \leq \frac{\|x\|}{1 - \|x\|}$$

<u>Proof:</u> $y := \sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n$

exists in A, since $\left(\sum_{n=0}^{k} x^n\right)_k$ is a Cauchy sequence: for $k > l$

$$\left\|\sum_{n=0}^{k} x^n - \sum_{n=0}^{l} x^n\right\| = \left\|\sum_{n=l+1}^{k} x^n\right\|$$

$$\leq \sum_{n=l+1}^{k} \|x\|^n < \varepsilon$$

for $l, k \geq N(\varepsilon)$

since $\left(\sum_{n=0}^{k} r^n\right)_k$ for $r = \|x\| < 1$ is Cauchy sequence

$y = (1-x)^{-1}$, since

$$\left(\sum_{n=0}^{k} x^n\right)(1-x) = 1 - x^{k+1} = (1-x)\left(\sum_{n=0}^{k} x^n\right)$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \quad k\to\infty \\ y & & 0 & & y \end{array}$$

$$\Rightarrow y(1-x) = 1 = (1-x)y$$

$$\|y\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \le \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|}$$

$$1 - y = -\sum_{n=1}^{\infty} x^n$$

$$\Rightarrow \|1-y\| \le \sum_{n=1}^{\infty} \|x\|^n = \frac{\|x\|}{1-\|x\|}$$

□

<u>12.7. Proposition</u>: Let A be a Banach algebra. Then the group $\mathcal{G}_A$ of invertible elements is open, and the mapping $x \mapsto x^{-1}$ is continuous on $\mathcal{G}_A$.

<u>Proof</u>: Consider $a \in \mathcal{G}_A$

to show: $b \in \mathcal{G}_A$ for $b \in A$ with $\|b-a\|$ sufficiently small

Assume $\|b-a\| < \frac{1}{\|a^{-1}\|}$

$$\Rightarrow b = a - (a-b)$$
$$= \underbrace{\left(1-(a-b)a^{-1}\right)}_{\text{invertible, since}} a$$

$\|(a-b)a^{-1}\| \leq \|a-b\| \cdot \|a^{-1}\| < 1$

$$\Rightarrow b^{-1} = a^{-1}\left(1-(a-b)a^{-1}\right)^{-1}$$

$$\Rightarrow b \in \mathcal{G}_A$$

Consider now: $x_n, x \in \mathcal{G}_A$ with $x_n \to x$

to show: $x_n^{-1} \to x^{-1}$

first: $x = 1$, i.e., $x_n \to 1$

to show: $x_n^{-1} \to 1^{-1} = 1$

$$\|1 - x_n^{-1}\| = \|1 - (1-(1-x_n))^{-1}\|$$

$$\leq \frac{\|1-x_n\|}{1-\|1-x_n\|}$$ by 12.6. for n s.th. $\|1-x_n\| < 1$

$$\xrightarrow{n\to\infty} 0$$

thus: $x_n^{-1} \to 1$

now general $x_n \to x$:

$$\Rightarrow \quad x^{-1} x_n \to 1$$

$$\overset{\text{above}}{\Rightarrow} \quad \underbrace{(x^{-1} x_n)^{-1}}_{\substack{\| \\ x_n^{-1} \cdot x}} \to 1 \quad \Rightarrow \quad x_n^{-1} \longrightarrow x^{-1}$$

$\square$

<u>12.8. Lemma</u>: Let A be Banach algebra, $a \in A$ and $\lambda, \mu \in \rho(a)$. Then

i) $R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$, i.e.

$$\frac{1}{\lambda - a} - \frac{1}{\mu - a} = (\mu - \lambda) \frac{1}{\lambda - a} \cdot \frac{1}{\mu - a}$$

resolvent equation

ii)
$$\lim_{\mu \to \lambda} \frac{1}{\lambda - \mu} \left(R(\lambda) - R(\mu) \right) = - R(\lambda)^2$$

<u>Proof</u>: i) $R(\lambda) = R(\lambda)(\mu - a) R(\mu)$

$$= R(\lambda)((\lambda - a) + (\mu - \lambda)) R(\mu)$$

$$= R(\mu) + R(\lambda)(\mu - \lambda) R(\mu)$$

ii) $\frac{1}{\lambda - \mu} (R(\lambda) - R(\mu)) \overset{i)}{=} - R(\lambda) R(\mu) \xrightarrow{\mu \to \lambda} - R(\lambda)^2$

$\square$

12.9. Theorem: Let A be Banach algebra and $a \in A$. Then $\sigma(a)$ is compact and non-empty and

$$\sigma(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$$

12.10. Remarks: 1) Main statement is

$$\sigma(a) \neq \emptyset$$

Note that for $A = M_n(\mathbb{C})$ this means: every matrix T has at least one eigenvalue, i.e., the equation $\det(\lambda 1 - T) = 0$ has at least one solution

$\rightarrow$ fundamental theorem of algebra

2) Of course, for this it is crucial to work over $\mathbb{C}$. For $F = \mathbb{R}$, it can happen that the spectrum is empty.

e.g.: $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ has $\sigma(a) = \{+i, -i\}$

but for $a \in M_2(\mathbb{R})$ the corresponding spectrum is empty!

$\sigma_{\mathbb{R}}(a) \subset \mathbb{R}$

<u>Proof of 12.9.</u>: 12.7. implies

$\lambda - a$ invertible $\Rightarrow$ $\mu - a$ invertible if

$|\lambda - \mu|$ sufficiently small

$\Rightarrow \rho(a)$ open

$\Rightarrow \sigma(a) = \mathbb{C} \setminus \rho(a)$ closed

Now consider $|\lambda| > \|a\|$

$$\Rightarrow (\lambda \cdot 1 - a)^{-1} = \frac{1}{\lambda} \underbrace{\left(1 - \frac{a}{\lambda}\right)^{-1}}$$

exists, by 12.6., since

$$\left\|\frac{a}{\lambda}\right\| = \frac{\|a\|}{|\lambda|} < 1$$

$\Rightarrow \lambda \in \rho(a)$

$\Rightarrow \sigma(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$

thus: $\sigma(a)$ is closed and bounded,

i.e., $\sigma(a)$ compact

remains to show: $\sigma(a) \neq \emptyset$

Assume $\sigma(a) = \emptyset$, i.e., $\rho(a) = \mathbb{C}$

$\Rightarrow z \mapsto R(z) = \dfrac{1}{z - a}$ is defined on $\mathbb{C}$

[idea: $R(z)$ "holomorphic", $\lim_{z \to \infty} R(z) = 0 \overset{\text{Liouville!?}}{\Longrightarrow} R(z) = 0$]

We have to go from operator-valued function $R(z)$ to a scalar-valued one by applying linear functional

$0 \in \rho(a) \Rightarrow a^{-1}$ exists and $a^{-1} \neq 0$

$\xRightarrow[\text{Banach}]{\text{Hahn}} \exists L \in A^*$ s.th. $L(a^{-1}) \neq 0$

Define now $f: \mathbb{C} \to \mathbb{C}$ by

$$f(z) := L(R(z)) = L\left(\frac{1}{z-a}\right)$$

then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= L\left(\lim_{\Delta z \to 0} \frac{R(z+\Delta z) - R(z)}{\Delta z}\right)$$

$$= L(-R(z)^2) \qquad \text{by 12.8 (ii)}$$

$\Rightarrow f'(z)$ exists for all $z \in \mathbb{C}$, thus

f holomorphic on $\mathbb{C}$, i.e.

f entire function

furthermore: $\lim_{z \to \infty} f(z) = 0$, since we have for $|z| > \|a\|$:

$$\|R(z)\| = \left\| \frac{1}{z-a} \right\|$$

$$= \frac{1}{|z|} \left\| \frac{1}{1 - \frac{a}{|z|}} \right\|$$

$$\overset{12.6}{\leq} \frac{1}{|z|} \cdot \frac{1}{1 - \frac{\|a\|}{|z|}} = \frac{1}{|z| - \|a\|}$$

$$\xrightarrow{z \to \infty} 0$$

thus $R(z) \xrightarrow{z \to \infty} 0$

$\Longrightarrow \underbrace{L(R(z))}_{f(z)} \longrightarrow L(0) = 0$ since $L \in A^*$ bounded

thus: $\lim\limits_{z \to \infty} f(z) = 0$

$\Longrightarrow f$ bounded on $\mathbb{C}$, and entire

$\overset{\text{Liouville}}{\Longrightarrow}$ f constant $= 0 \Longrightarrow f \equiv 0$

but then: $f(0) = L(R(0)) = L(a^{-1}) \neq 0$

contradiction

$\Longrightarrow \sigma(a) \neq \emptyset$

$\square$

12.10. Corollary (Theorem of Gelfand-Mazur):

Let A be a Banach algebra in which each element $a \neq 0$ is invertible. Then $A \cong \mathbb{C}$.

Proof: $a \in A \overset{12.9.}{\Longrightarrow} \sigma(a) \neq \emptyset$, i.e.

$\exists\, \lambda \in \sigma(a)$: $\lambda 1 - a$ not invertible

$\Longrightarrow \lambda 1 - a = 0$

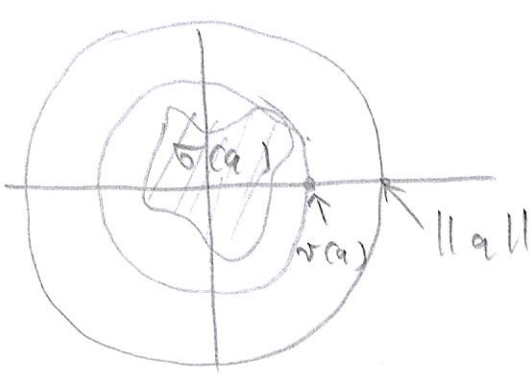
$\Longrightarrow \lambda \cdot 1 = a$ $\square$

12.11. Remark: This implies that the quaternions cannot be equipped with a norm that makes it to a Banach algebra.

12.12. Def: Let A be Banach algebra and $a \in A$. Then

$$r(a) := \sup\{ |\lambda| : \lambda \in \sigma(a) \}$$

is the spectral radius of a.



12.13. Theorem (spectral radius formula):

1) Let A be a Banach algebra and $a \in A$. Then

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

2) If A is a C^*-algebra and $a \in A$ normal (i.e., $aa^* = a^*a$), then

$$r(a) = \|a\|$$

12.14. Remarks: 1) Note: $r(a)$ is defined in terms of the algebra structure of A, whereas the right hand depends on the norm/topology.

2) In general it can happen that $r(a) < \|a\|$; for example: $a \neq 0$ with $\sigma(a) = \{0\}$

$$\Rightarrow r(a) = 0, \text{ but } \|a\| \neq 0$$

e.g. $a =$ Volterra operator or

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Proof of 12.13.: 1) note: $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$, since

$$\lambda^n - a^n = (\lambda - a)(\lambda^{n-1} + \lambda^{n-2}a + \dots + \lambda a^{n-2} + a^{n-1})$$

thus: $\lambda^n - a^n$ invertible $\Rightarrow$ $\lambda - a$ invertible with

$$(\lambda - a)^{-1} = (\lambda^{n-1} + \dots + a^{n-1})(\lambda^n - a^n)^{-1}$$

hence:

$$\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n) \overset{12.1.}{\Longrightarrow} |\lambda|^n \leq \|a^n\| \quad \forall n$$

$$\Rightarrow |\lambda| \leq \liminf \|a^n\|^{1/n} \quad \forall \lambda \in \sigma(a)$$

$$\Rightarrow r(a) \leq \liminf \|a^n\|^{1/n}$$

remains to show: $r(a) \geq \limsup \|a^n\|^{1/n}$

Idea: $R(z) = \frac{1}{z-a} = \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$ if $\|a\| < |z|$

power series expansion of $R(z)$ is valid for all circles where $R(z)$ is holomorph, i.e. for $|z| > r(a)$

classical formula for radius of convergence of power series $\sum \frac{\alpha_n}{z^{n+1}}$ is

$$\limsup |\alpha_n|^{1/n} \overset{?}{=} \limsup \|a^n\|^{1/n}$$

Problem: $R(z)$ is operator-valued, we have to reduce it to $\mathbb{C}$, by applying linear functional

Let $L \in A^*$, then put

$$f(z) := L(R(z)) = L\left(\frac{1}{z-a}\right)$$

$\Rightarrow f(z) : \mathcal{S}(a) \to \mathbb{C}$ holomorph and

$$f(z) = \sum_{n=0}^{\infty} \frac{L(a^n)}{z^{n+1}} \quad \text{for } |z| > \|a\|$$

$\Rightarrow$ this power series expansion is also valid for all $|z| > r(a)$, thus

$$\limsup |L(a^n)|^{1/n} \leq r(a)$$

Consider now $r > r(a)$, then

$$|L(a^n)|^{1/n} \leq r \quad \text{for } n \text{ sufficiently large}$$

$$\Rightarrow \frac{|L(a^n)|}{r^n} \leq 1 \qquad \text{———} \;"\; \text{———}$$

$$\Rightarrow \sup_{n \in \mathbb{N}} \frac{|L(a^n)|}{r^n} < \infty \qquad \forall L \in A^*$$

$\underset{\text{Boundedness 7.6.}}{\overset{\text{Uniform}}{\Longrightarrow}}$

$$\sup_{n \in \mathbb{N}} \left\| \frac{a^n}{r^n} \right\| < \infty$$

$$\sup_{x \in \mathcal{M}} |L(x)| < \infty \;\; \forall L \in A^* \Rightarrow \sup_{x \in \mathcal{M}} \|x\| < \infty$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\sup_{\hat{x} \in \hat{\mathcal{M}}} |\hat{x}(L)| < \infty \;\; \forall L \in A^* \Rightarrow \sup_{\hat{x} \in \hat{\mathcal{M}}} \|\hat{x}\| < \infty$$

$$\Rightarrow \exists\, C' : \left\| \frac{a^n}{r^n} \right\| \leq C' \quad \forall n$$

i.e., $\|a^n\| \leq C' r^n$

$$\Rightarrow \quad \|a^n\|^{1/n} \leq C'^{1/n} \cdot r \xrightarrow{n \to \infty} r$$

$$\Rightarrow \limsup \|a^n\|^{1/n} \leq r \quad \forall r > r(a)$$

$$\Rightarrow \limsup \|a^n\|^{1/n} \leq r(a)$$

and thus:

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

2) Let A be C^*-algebra, and $aa^* = a^*a$

$$\Rightarrow \|a^2\|^2 = \|(a^2)^* a^2\|$$

$$= \| a^* \underbrace{a^* a}_{a a^*} a \|$$

$$= \|(a^*a)^*(a^*a)\|$$

$$= \|a^*a\|^2$$

$$= \|a\|^4$$

$$\Rightarrow \|a^2\| = \|a\|^2$$

induction $\Longrightarrow$ $\|a^{2^n}\| = \|a\|^{2^n}$ (note: a normal $\Rightarrow a^k$ normal)

thus: $$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

$$= \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n}$$

$$= a$$ $\square$

Consider now $a \in B \subset A$

A, B Banach algebras

In general, $\sigma(a)$ will depend on the algebra, i.e. $\sigma_A(a) \subset \sigma_B(a)$

$\uparrow$ but "$\neq$" possible

<u>12.15. Example:</u> Let

$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ and

$\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$

Put $A := C(\mathbb{T})$

$B := \overline{\{p : \mathbb{T} \to \mathbb{C} \mid p \text{ polynom in } z\}}^{\|\cdot\|}$

note: $B \subsetneq A = \overline{\{p : \mathbb{T} \to \mathbb{C} \mid p \text{ polynom in } z \text{ and } \bar{z}\}}^{\|\cdot\|}$

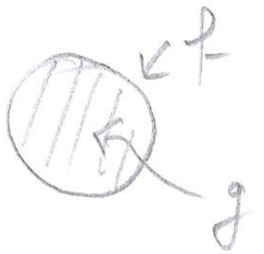
$\uparrow$ Stone-Weierstraß

Consider now $p \in B \subset A$, where $p(z) = z$

$\Rightarrow \sigma_A(p) = \mathbb{T} = \partial D$

$\sigma_B(p) = \, ?$

note: $f \in B \xRightarrow[\text{analysis}]{\text{complex}} f$ can be extended to holomorphic (in particular, continuous) function g on D, s.th. $g|\partial D = f$



Consider now $\lambda \notin \sigma_B(p)$

$\Rightarrow \exists f \in B : (\lambda - z) f(z) = 1$ on $\mathbb{T}$

this extends to all of D

$\Rightarrow \exists g \in \mathcal{C}(D)$, $(g|\partial D = f)$ s.th.

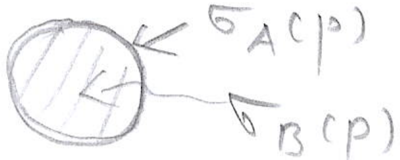
$(\lambda - z) g(z) = 1 \quad \forall z \in D$

$\Rightarrow \lambda \neq z \quad \forall z \in D$

$\Rightarrow D \subset \sigma_B(p)$

Since $\sigma_B(p) \subset \{\lambda : |\lambda| \le \underbrace{\|p\|}_{=1}\} = D$

$\Rightarrow \sigma_B(p) = D$ thus:



12.16. Theorem: Let A, B be Banach algebras with $B \subset A$ and consider $a \in B$. Then:

i) $\sigma_A(a) \subset \sigma_B(a)$

ii) $\partial \sigma_B(a) \subset \partial \sigma_A(a)$

Proof: i) $\lambda \notin \sigma_B(a) \Rightarrow \exists (\lambda - a)^{-1} \in B \subset A$

$$\Rightarrow \lambda \notin \sigma_A(a)$$

ii) Consider $\lambda \in \partial \sigma_B(a)$

to show: $\lambda \in \partial \sigma_A(a) = \sigma_A(a) \setminus \mathring{\sigma}_A(a)$

$$\left.\begin{array}{l} \text{Since } \mathring{\sigma}_A(a) \subset \mathring{\sigma}_B(a) \\ \lambda \notin \mathring{\sigma}_B(a) \end{array}\right\} \Rightarrow \lambda \notin \mathring{\sigma}_A(a)$$

remains to show: $\lambda \in \sigma_A(a)$

Assume $\lambda \notin \sigma_A(a)$, i.e. $\exists x \in A$ with

$$x(\lambda - a) = (\lambda - a)x = 1$$

$\lambda \in \partial \sigma_B(a) \Rightarrow \exists \lambda_n \in \mathbb{C} \setminus \sigma_B(a)$ with $\lambda_n \to \lambda$

$$(\lambda_n - a)^{-1} \in B \subset A$$

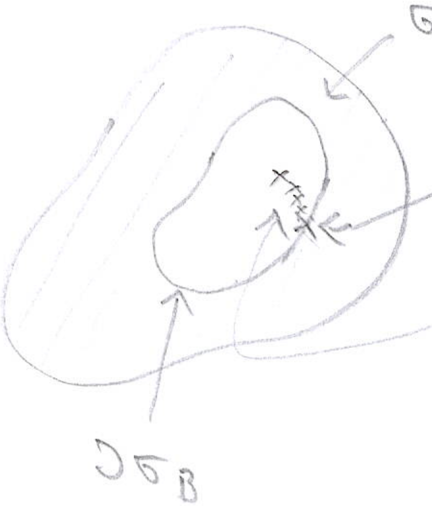
$\lambda_n \to \lambda \Rightarrow \lambda_n - a \to \lambda - a$

$$\overset{\text{inversion}}{\underset{\text{continuous}}{\Longrightarrow}} \underbrace{(\lambda_n - a)^{-1}}_{\in B} \to (\lambda - a)^{-1} = x$$

B complete $\Rightarrow \underset{\substack{\| \\ (\lambda-a)^{-1}}}{x} \in B \quad \Rightarrow \lambda \notin \sigma_B(a)$

contradiction

$\Rightarrow \lambda \in \sigma_A(a)$



$\lambda \in \partial\sigma_B$, but $\lambda \notin \sigma_A$

$\Downarrow$

$\exists \lambda_n \in \rho_B$, i.e.

$(\lambda_n - a)^{-1} \in B \xrightarrow{n \to \infty} (\lambda - a)^{-1} \in A$

$\square$

<u>12.17. Theorem:</u> Let A be a commutative C^*-algebra and $a \in A$ selfadjoint, i.e. $a^* = a$. Then $\sigma(a) \subseteq \mathbb{R}$

<u>Proof:</u> later! (in 13.8)

<u>12.18. Theorem:</u> Let A and B be C^*-algebras with $B \subset A$ and $a \in B$. Then

$$\sigma_A(a) = \sigma_B(a)$$

12.19. Remark: For $a = a^* \in A$ put

$$B := C^*(a,1) := \overline{\{ \alpha_0 \cdot 1 + \alpha_1 a + \dots + \alpha_n a^n \mid {n \in \mathbb{N} \atop \alpha_i \in \mathbb{C}} \}}$$

$$\subset A,$$

this is the C^*-algebra generated by a (and 1).

$\Rightarrow$ B is commutative, thus

$$\sigma_A(a) \overset{12.18}{=} \sigma_B(a) \overset{12.17}{\subset} \mathbb{R}$$

i.e. 12.17 is also true for arbitrary C^*-algebra.

Proof of 12.18: Consider $\overset{\text{first}}{a} = a^* \in B$, put

$C := C^*(a,1)$ commutative C^*-algebra

and $$C \subset B \subset A$$

We show: $\sigma_A(a) = \sigma_C(a) = \sigma_B(a)$

$$C \subset B \overset{12.16}{\Longrightarrow} \sigma_B(a) \subset \sigma_C(a) = \partial\sigma_C(a) \subset \partial\sigma_B(a)$$

since $\sigma_C(a) \subset \mathbb{R}$

$$\subset \sigma_B(a)$$

$$\left.\begin{array}{l} \Rightarrow \sigma_B(a) = \sigma_C(a) \\ \text{in the same way: } \sigma_A(a) = \sigma_C(a) \end{array}\right\} \Rightarrow \sigma_A(a) = \sigma_B(a)$$

Consider now arbitrary $a \in B$:

it suffices to show: a invertible in A

$\Rightarrow$ a invertible in B

So let a be invertible in A, i.e.

$\exists\, b \in A : ab = 1 = ba$

$\Rightarrow b^* a^* = 1 = a^* b^* \qquad (1^* = 1)$

$\Rightarrow (a^* a)(bb^*) = 1 = (bb^*)(a^* a)$

thus: $a^* a$ is invertible in A

$(a^* a)^* = a^* a$ is selfadjoint

$\overset{\text{above}}{\Longrightarrow}$ $a^* a$ is invertible in B and

$$bb^* = (a^* a)^{-1} \in B$$

$\Rightarrow b = b(b^* a^*) = \underbrace{(bb^*)}_{\in B} a^* \in B$

$\Rightarrow a^{-1} = b \in B$ $\qquad \square$