12. Banach Algebras

in the following $\mathbb{F} = \mathbb{C}$!

<u>12.1. Def.</u>: 1) An <u>algebra</u> is a vector space A with a multiplication $\cdot : A \times A \to A$, such that $(A, +, \cdot)$ is a ring with unit 1 and

$$\lambda(ab) = (\lambda a) b = a(\lambda b) \quad \forall a, b \in A, \ \lambda \in \mathbb{C}$$

If $ab = ba \ \forall a, b \in A$, then A is <u>commutative</u>.

2) A <u>normed algebra</u> is an algebra A, equipped with a norm $\|\cdot\|$, such that

i) $\|ab\| \leq \|a\| \cdot \|b\| \quad \forall a, b \in A$

ii) $\|1\| = 1$

3) A <u>Banach algebra</u> is a normed algebra which is complete.

4) A <u>C^*-algebra</u> is a Banach algebra with a mapping $*: A \to A$ such that:

i) $*$ is an involution, i.e. for $a, b \in A$, $\lambda \in \mathbb{C}$

$$(a+b)^* = a^* + b^*$$

$$(\lambda a)^* = \bar{\lambda}\, a^*$$

$$(ab)^* = b^* a^*$$

$$(a^*)^* = a$$

ii) $\| a^* a \| = \| a \|^2 \quad \forall a \in A$

5) Let A_1, A_2 be Banach algebras and $\varphi: A_1 \to A_2$ a linear map.

If φ is multiplicative, i.e,

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in A_1,$$

then φ is an (algebra) <u>homomorphism</u>.

If φ is bijective, then it is an (algebra) <u>isomorphism</u>.

If furthermore $\|\varphi(a)\| = \|a\| \quad \forall a \in A_1$ then φ is an <u>isometric isomorphism</u>.

6) Let A_1, A_2 be C^*-algebras and $\varphi: A_1 \to A_2$ a homomorphism. If

$$\varphi(a^*) = \varphi(a)^* \quad \forall a \in A_1$$

then φ is a <u>$*$-homomorphism</u>.

<u>12.2. Remark:</u> One can also consider Banach algebras without unit. Those can be embedded in Banach algebras with unit. (Exercise!)

<u>12.3. Examples:</u> 1) $K \neq \emptyset$ compact space

$$C(K) := \{ f: K \to \mathbb{C} \mid f \text{ continuous} \}$$

is Banach algebra with

$$\|f\| := \sup_{t \in K} |f(t)|$$

$$\left.\begin{aligned} (f+g)(t) &:= f(t) + g(t) \\ (\lambda \cdot f)(t) &:= \lambda f(t) \\ (f \cdot g)(t) &:= f(t) \cdot g(t) \end{aligned}\right\} \quad \begin{aligned} f, g &\in C(K) \\ \lambda &\in \mathbb{C} \end{aligned}$$

$C(K)$ is a <u>C^*-algebra</u> with

$$f^*(t) := \overline{f(t)}$$

$C(K)$ is commutative

2) Let X be Banach space. Then $B(X)$ is a Banach algebra; it is non-commutative, if $\dim X > 1$

3) Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a C^*-algebra.

<u>12.4. Lemma:</u> 1) Let A be a normed algebra. Then the multiplication is continuous.

2) Let A be a C^*-algebra. Then

i) $\|a^*\| = \|a\| \quad \forall a \in A$

ii) $1^* = 1, \quad 0^* = 0$

<u>Proof:</u> 1) $\|ab - cd\| = \|a(b-d) + (a-c)d\|$

$$\leq \|a\| \, \|b-d\| + \|a-c\| \, \|d\|$$

thus: $\left.\begin{array}{l} a_n \to a \\ b_n \to b \end{array}\right\} \Rightarrow a_n b_n \longrightarrow ab$

2) i) $\|a\|^2 = \|a a^*\| \leq \|a\| \cdot \|a^*\|$

$\Rightarrow \|a\| \leq \|a^*\|$ (assume w.v. $a \neq 0$)

$\overset{a \mapsto a^*}{\Rightarrow} \|a^*\| \leq \|a^{**}\| = \|a\|$

$\Rightarrow \|a\| = \|a^*\|$

ii) $0^* = (0+0)^* = 0^* + 0^* \Rightarrow 0^* = 0$

$$1^* = 1 \cdot 1^*$$

$$\Rightarrow 1 = 1^{**} = (1 \cdot 1^*)^* = 1^{**} \cdot 1^* = 1 \cdot 1^* = 1^*$$

□

12.5. Def.: Let A be a Banach algebra.

1) $a \in A$ is invertible, if there exists $b \in A$ such that $a \cdot b = 1 = b \cdot a$

We denote b then by a^{-1}.

(Note: b is uniquely determined.)

$\mathcal{G}_A := \{a \in A \mid a \text{ is invertible}\}$

is the group of invertible elements.

(group w.r.t. multiplication:

$x, y \in \mathcal{G}_A \Rightarrow (xy)^{-1} = y^{-1}x^{-1} \Rightarrow xy \in \mathcal{G}_A$)

2) $\varrho(a) := \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ is invertible}\}$

is the resolvent set of a. The function

$R(\cdot, a) : \varrho(a) \to A$, $R(\lambda, a) := (\lambda 1 - a)^{-1}$
$=: R(\lambda)$

is the resolvent of a

(in physics also: Greens function $G(z)$)

The complement of $\rho(a)$,

$$\sigma(a) := \mathbb{C} \setminus \rho(a)$$
$$= \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ not invertible}\}$$

is the <u>spectrum</u> of a.

<u>12.6. Lemma:</u> Let A be a Banach algebra and $x \in A$ with $\|x\| < 1$. Then $1 - x$ is invertible and we have

$$\|(1-x)^{-1}\| \leq \frac{1}{1 - \|x\|}$$

$$\|1 - (1-x)^{-1}\| \leq \frac{\|x\|}{1 - \|x\|}$$

<u>Proof:</u> $y := \sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n$

exists in A, since $\left(\sum_{n=0}^{k} x^n\right)_k$ is a Cauchy sequence: for $k > l$

$$\left\|\sum_{n=0}^{k} x^n - \sum_{n=0}^{l} x^n\right\| = \left\|\sum_{n=l+1}^{k} x^n\right\|$$

$$\leq \sum_{n=l+1}^{k} \|x\|^n < \varepsilon$$

for $l, k \geq N(\varepsilon)$

since $\left(\sum_{n=0}^{k} r^n\right)_k$ for $r = \|x\| < 1$ is Cauchy sequence

$y = (1-x)^{-1}$, since

$$\left(\sum_{n=0}^{k} x^n\right)(1-x) = 1 - x^{k+1} = (1-x)\left(\sum_{n=0}^{k} x^n\right)$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \quad k \to \infty \\ y & & 0 & & y \end{array}$$

$$\Rightarrow y(1-x) = 1 = (1-x)y$$

$$\|y\| = \left\|\sum_{n=0}^{\infty} x^n\right\| \le \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1-\|x\|}$$

$$1 - y = -\sum_{n=1}^{\infty} x^n$$

$$\Rightarrow \|1-y\| \le \sum_{n=1}^{\infty} \|x\|^n = \frac{\|x\|}{1-\|x\|}$$

□

<u>12.7. Proposition:</u> Let A be a Banach algebra. Then the group $\mathcal{G}_A$ of invertible elements is open, and the mapping $x \mapsto x^{-1}$ is continuous on $\mathcal{G}_A$.

<u>Proof:</u> Consider $a \in \mathcal{G}_A$

to show: $b \in \mathcal{G}_A$ for $b \in A$ with $\|b-a\|$ sufficiently small

Assume $\|b-a\| < \frac{1}{\|a^{-1}\|}$

$$\Rightarrow b = a - (a-b)$$
$$= \underbrace{\left(1 - (a-b)\,a^{-1}\right)}_{\text{invertible, since}}\, a$$

$\|(a-b)\,a^{-1}\| \leq \|a-b\| \cdot \|a^{-1}\| < 1$

$\Rightarrow b^{-1} = a^{-1}\left(1-(a-b)\,a^{-1}\right)^{-1}$

$\Rightarrow b \in \mathcal{G}_A$

Consider now: $x_n, x \in \mathcal{G}_A$ with $x_n \to x$

to show: $x_n^{-1} \to x^{-1}$

first: $x = 1$, i.e., $x_n \to 1$

to show: $x_n^{-1} \to 1^{-1} = 1$

$$\|1 - x_n^{-1}\| = \|1 - (1-(1-x_n))^{-1}\|$$

$$\leq \frac{\|1-x_n\|}{1-\|1-x_n\|}$$ by 12.6. for n s.th. $\|1-x_n\| < 1$

$$\xrightarrow{n\to\infty} 0$$

thus: $x_n^{-1} \to 1$